State Observability and Observers of Linear-Time-Invariant Systems under Irregular Sampling and Sensor Limitations

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***Abstract*—State observability and observer designs are investigated for linear-time-invariant systems in continuous time when the outputs are measured only at a set of irregular sampling time sequences. The problem is primarily motivated by systems with limited sensor information in which sensor switching generates irregular sampling sequences. State observability may be lost and the traditional observers may fail in general, even if the system has a full-rank observability matrix. It demonstrates that if the original system is observable, the irregularly sampled system will be observable if the sampling density is higher than some critical frequency, independent of the actual time sequences. This result extends Shannon's sampling theorem for signal reconstruction under periodic sampling to system observability under arbitrary sampling sequences. State observers and recursive algorithms are developed whose convergence properties are derived under potentially dependent measurement noises. Persistent excitation conditions are validated by designing sampling time sequences. By generating suitable switching time sequences, the designed state observers are shown to be convergent in mean square, with probability one, and with exponential convergence rates. Schemes for generating desired sampling sequences are summarized.**

***Index Terms*—Observability, state observers, irregular sampling, persistent excitation, quantized sensors, strong convergence, mean square convergence.**

I. Introduction

State estimation of linear-time-invariant systems is studied when output observations are sampled on a set of non-periodic and irregular sampling times and the sampled values are corrupted by possibly dependent noises. State observability may be lost and the traditional observers may fail in general, even if the system has a full-rank observability matrix. This paper contributes to this problem in several related aspects: (1) Observability of systems under arbitrary sampling time sequences is established. (2) An observer design procedure is introduced and recursive algorithms are developed. (3) The critical issue of persistent excitation conditions for convergence of state estimation under irregular sampling is resolved. (4) Convergence properties and convergence rates of the designed state observers are derived.

Irregular sampling time sequences may be generated passively due to event-triggered sampling or low-resolution signal quantization, or actively by input control or threshold adaptation under binary-valued sensors. While synchronized periodic sampling has been extensively studied, non-uniform sampling has emerged from many applications. Studies on fundamental properties of non-uniform sampling remain an active area of research, see [9] and the references therein for some recent work in this area. Our problem here was initially investigated in [35]. The current paper demonstrates that if the continuous-time system is observable, then within any finite time interval, there exists a critical frequency such that as long as the density of sampling points exceeds the frequency, the initial state can be uniquely reconstructed, regardless the actual sampling time sequences. In this sense, it resolves completely the issue of observability of sampled systems with arbitrary sampling times. The result can be compared to the Nyquist frequency of a signal in Shannon's sampling theorem under synchronized sampling for signal reconstruction.

Binary sensors are used in many practical systems and systems involving communication channels. Due to sensor nonlinearity, the state estimation becomes a nonlinear filtering problem. The added difficulty is that the nonlinearity is non-smooth and has only two output values. As a result, methodologies that rely on local linearization or small perturbation analysis such as extended Kalman filters are not applicable. Also, under irregular sampling, the sampled systems are time varying, and cannot be lifted to become linear-time invariant systems. Observer design methods for linear-time-invariant systems such as deadbeat design, pole placement, H^2/H^∞ filtering [6], [8], [16], [28], etc., cannot be directly applied. Early work on optimal nonlinear filtering can be traced back to Stratonovich [30]. The first rigorous derivation was given by Kushner in his seminal paper [19]; see also [20]. Such equations are referred to as Kushner's equations. Subsequently, the conditional law of the filter was further developed by Zakai [39], later referred to as Zakai's equations. Nowadays, one of the well-known stochastic PDEs for nonlinear filters is the so-called Duncan-Mortensen-Zakai equation. One main challenge faced by nonlinear filtering problems is that the filters are

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generally infinite dimensional. Much effort has been devoted to finding finite dimensional filters. However, to date, there are only a handful of finite dimensional nonlinear filters known in existence, including the Wonham filter, Benes filter, etc.

To deal with nonlinear and time-varying systems and random noises, we use a time domain formulation in a stochastic framework. The main challenges in these problems include: (1) When will the sampled system be observable under irregular sampling sequences? (2) How can the states be estimated on the basis of irregular sampling points? (3) How should the set of sampling times be selected to ensure that persistent excitation conditions for state estimation are validated? (4) What convergence properties can be derived for the state estimates? This paper focuses on resolving these issues. Since this paper deals with convergence analysis in stochastic systems, it is related to many classical treatments of similar topics. We cite [12], [13], [15], [21], [29] for their relevance to this paper, but emphasize that state observability under irregular sampling, state estimation with binary sensors, active observer design, convergence under mixing-type sampling disturbances, and convergence rates of state observers under irregular sampling schemes mark departure of our results from traditional ones.

The rest of the paper is organized as follows. Section II formulates the state observation problems pursued in this paper. Section III focuses on observability of systems under arbitrary sampling times. The main result (Theorem 1) resolves this issue completely. The observer design is studied in Section IV. An observer structure is introduced first, which includes a discrete state estimator and a continuous-time state observer with updating schemes. It is shown that if the original system is observable, then the observer can be constructed and implemented. Recursive algorithms are derived. After certain normalization steps in expressing estimation errors, Section V investigates conditions on sampling time sequences that will validate persistent excitation conditions for convergence of state estimation. Section VI is devoted to convergence analysis. Subsection VI-A establishes convergence properties of the state observer in a finite horizon. Mean-squares (MS) convergence and strong (with probability one) convergence are established. Subsection VI-B deals with convergence analysis of the state observer over an infinite horizon. By generating suitable switching time sequences, we show that not only MS and strong convergence properties can be guaranteed, but also the convergence is exponentially fast. Generation of irregular sampling-time sequences is discussed in Section VII, including passive types of signal quantization and event-triggered sampling, and strategies of binary-valued sensors with active input control or threshold adaptation. An illustrative example is provided in Section VIII. Finally, some concluding remarks are given in Section IX.

Some related results on identification, state estimation, and fault detection using binary or quantized outputs can be found in [17], [31], [33], [34], [35], [36]. In relation to the existing knowledge on observability of sampled systems, we refer the reader to standard textbooks on digital control systems; see, e.g., [7], [18], [24] for classical synchronized sampling schemes on linear systems, and [1], [11] on nonlinear systems. Shannon's sampling theorem is a fundamental result on digital signal processing, see [25]. References [26], [31] contain more recent studies on observability of sampled systems. Irregular sampling may occur due to event triggered sampling [3], [23] or communication uncertainty and interruptions [12]. Some preliminary results of this paper were reported in [37], [38].

II. Problem Formulation

Consider an MISO (multi-input-single-output) linear time-invariant system

$$\begin{cases} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t), \end{cases} \qquad t \geq 0 \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and for $t \geq 0$, $u(t) \in \mathbb{R}^m$ is the control input, $x(t) \in \mathbb{R}^n$ is the state, and $y(t) \in \mathbb{R}$ is the system output.[1] The initial state is unknown. We are interested in estimation $x(t)$, from some limited observations on $y(t)$.

In our setup, the output $y(t)$ is only measured at a sequence of sampling time instants $\{t_i\}$ with measured values $\gamma(t_i)$[2] and noise d_i

$$\gamma(t_i) = y(t_i) - d_i. \tag{2}$$

We would like to estimate the state $x(t)$ from information on $u(t)$, $\{t_i\}$ and $\{\gamma(t_i)\}$. In general, the sampling-time sequence is irregular and non-periodic. In this paper, the switching time sequence will be deterministic. Generation of such irregular sampling-time sequences may be passive as in event-triggered sampling and low-resolution signal quantization. Here, we are mostly motivated by state estimation problems with binary-valued observations. Active control of inputs or thresholds causes system outputs to cross sensor thresholds and the sensor outputs to switch values, generating an irregular sampling-time sequence. As a result, the sequence $\{t_i\}$ will be interchangeably called "sampling-time sequence" or "switching-time sequence." Generation of such sequences will be discussed in Section VII.

It is obvious that state estimation will not be possible if the system is not observable. Also, in this paper, d_k is allowed to be dependent. These are stated in the following assumptions.

Assumption 1: The following conditions hold.

(a) The system is observable, i.e., $W_o' = [C' \ (CA)' \dots (CA^{n-1})']$ is full rank.
(b) The sensor noise $\{d_k\}$ is a stationary sequence of ρ^*-mixing random variables and $Ed_k = 0$.

Remark 1: We refer the reader to [32, p. 101] and [2, p. 1466] for a definition of ρ^* mixing processes. Mixing processes are essentially those whose remote past and distant future are asymptotically independent; different types of mixing processes are characterized by their mixing measures. Loosely speaking, a ρ^* mixing process is characterized by using covariance as a mixing measure. The class of ρ^* mixing processes include i.i.d. sequences, stationary martingale difference sequences, and certain ϕ-mixing processes [10, Section

[1]We limit our studies to MISO systems for notational simplicity, although technically inclusion of MIMO cases is quite straightforward. An MIMO system provides more measurement information than MISO systems. Consequently, if one output is sufficient for state estimation, so will be its MIMO counterpart.

[2]Using the notation $\gamma(t_i)$ is non standard, but consistent with the interpretation that it is the threshold value of a binary sensor.

7.2]. For example, a sequence of i.i.d. random variables $\{d_j\}$ (discrete-time 'white noise') is ρ^*-mixing with covariance $Ed_id_j = 0$ for $i \neq j$. A moving-averaging process of order m driven by a 'white noise' is also a ρ^*-mixing process, where the covariance becomes 0 for any time lag larger than m. Likewise, if d_k is a discrete-time Markov chain that is irreducible and aperiodic with a finite state space, then it is also a ρ^*-mixing process. Here the covariance decays exponentially fast. In addition, ρ^*-mixing also includes Markov chains with countable spaces satisfying certain conditions, as well as many other infinitely correlated noises as long as their correlations decay sufficiently fast. We should emphasize that our results show that the same convergence properties and convergence rates for i.i.d. noises continue to hold under ρ^* mixing processes, as such they are not more conservative. For the problems of this paper, practical importance of ρ^* mixing processes stems from the fact that if an i.i.d. noise passes through a communication channel with finite memory, or infinite but stable decaying memory, the received signal will be corrupted by ρ^* mixing noises.

To simplify notation, in some proofs we adopt the convention of using a generic positive constant $\kappa > 0$ to represent unspecified positive constants. Hence, for any $a > 0$, $a\kappa = \kappa$. This simplifies expressions where actual values of some positive constants are irrelevant to the conclusions.

III. Observability of Irregularly Sampled Systems

We start with the system in (1) with noise-free observations (so, $y(t_i) = \gamma(t_i)$ in this section). The initial state $x(0)$ is unknown. Suppose that the output of the system is sampled at a set of N time instances $\mathcal{T}_N = \{t_i \geq 0, i = 1, \ldots, N\}$, generating the set of observations $\mathcal{Y}_N = \{y(t_i), i = 1, \ldots, N\}$. Observability of the sampled system deals with reconstruction of the state $x(t)$ from output observations $\mathcal{Y}_N$. Since the system contains no uncertainty and the input is known, this is equivalent to reconstruction of the initial state $x(0)$ from $\mathcal{Y}_N$.

Definition 1: (i) System (1) is said to be $\mathcal{T}_N$ *observable* if $x(0)$ can be uniquely determined from any observations $\mathcal{Y}_N$ on $\mathcal{T}_N$. (ii) For a given time interval $[0, T]$ and an integer N_T, the system is said to be N_T*-sample observable* if the system is $\mathcal{T}_{N_T}$ *observable* for any $\mathcal{T}_{N_T}$ with $0 \leq t_i \leq T$, $i = 1, \ldots, N_T$.

We use some examples from [7], [27] to illustrate the key issues involved in observability of the irregularly sampled system. Suppose the system contains complex eigenvalues. Then for some non-zero initial state $x(0)$, the output will be zero infinitely often, say, at $t_i, i = 1, 2, \ldots$. If these are the actual sampling times, the output samples will be zeros and provide no information about $x(0)$, rendering the sampled system unobservable with respect to such observation sets. On the other hand, it is well understood that for any given n-dimensional observable system, there exists a sufficiently small time interval $[0, T]$ such that if n sampling times $t_i \in [0, T]$, then the system is always $\mathcal{T}_n$ observable. In other words, the system is n-sample observable. However, this is not the case when T is not small. The main goal of this section is to establish a general result that resolves the issues illustrated in these examples.

The main result of this section is the following theorem. Suppose the matrix A in (1) has eigenvalues $\lambda_j, j = 1, \ldots, n$. Denote $\delta := \max_{1 \leq i,j \leq n}\{\Im(\lambda_i - \lambda_j)\}$, where $\Im(z)$ is the imaginary part of $z \in \mathbb{C}$. For a given time interval $[0, T]$, define

$$\mu_T = 2(n-1) + \frac{T}{2\pi}\delta. \tag{3}$$

Theorem 1: Suppose that Assumption 1 (a) is true. Given $T > 0$, if $N \geq \mu_T$, the system in (1) is N-sample observable.

Remark 2: This theorem asserts that for a given $T > 0$, if all sampling points are confined in $[0, T]$ and the number of samples exceeds μ_T, the initial state can always be uniquely determined from $\mathcal{Y}_N$, regardless the actual sampling times. In some sense, the condition on sampling in Theorem 1 is the weakest restriction on a sampling scheme to reconstruct $x(0)$. Asymptotically for large T, $\frac{\mu_T}{T} \approx \frac{\delta}{2\pi} := \mu$, which is independent of the system order n. One may compare this with the Nyquist frequency of Shannon's sampling theorem for signals. Suppose the largest imaginary part of the eigenvalues of A is $\omega_{\max}$. Then, $\delta = 2\ \omega_{\max}$ and $\mu = \frac{\omega_{\max}}{\pi}$ is precisely the Nyquist frequency in Hz, if one interprets $\omega_{\max}$ as the bandwidth of the system. The result here is more general since it does not require the sampling scheme to be periodic. In the special case of synchronized sampling and pure imaginary eigenvalues, the sampled values may be all zeros if the sampling frequency is below μ (see [27] for details), implying that the observability is lost. In other words, μ is an asymptotically tight bound for large T.

The remainder of this section is devoted to proving Theorem 1. Without loss of generality, we may focus only on the zero-input relationship $y(t) = Ce^{At}x(0)$. Under N sampling times $\mathcal{T}_N = \{t_i, i = 1, \ldots, N\}$, we have $\mathcal{Y}_N = \{y(t_i), i = 1, \ldots, N\}$ with $y(t_i) = Ce^{At_i}x(0), i = 1, \ldots, N$, which can be written as $Y_N = M_N x(0)$, where

$$Y_N = \begin{bmatrix} y(t_1) \\ \vdots \\ y(t_N) \end{bmatrix}; \quad M_N = \begin{bmatrix} Ce^{At_1} \\ \vdots \\ Ce^{At_N} \end{bmatrix}. \tag{4}$$

Consequently, the system is $\mathcal{T}_N$ observable if and only if M_N is full rank.

We express e^{At} in terms of the matrices $I, A, A^2, \cdots, A^{n-1}$, $e^{At} = \alpha_1(t)I + \alpha_2(t)A + \cdots + \alpha_n(t)A^{n-1}$, where $\alpha(t) = [\alpha_1(t), \ldots, \alpha_n(t)]'$ can be solved by the Lagrange-Hermite interpolation (when the eigenvalues of A are all distinct, the method is called the Sylvester interpolation) [14]. Suppose A has l distinct eigenvalues λ_i, $i = 1, \ldots, l$ of multiplicity m_i, respectively. Here, $\sum_{i=1}^{l} m_i = n$. Define the modes[3] of the A matrix by

$$\xi(t) = \left[e^{\lambda_1 t}, \cdots, \frac{t^{m_1-1}}{(m_1-1)!}e^{\lambda_1 t}, \cdots, e^{\lambda_l t}, \cdots, \frac{t^{m_l-1}}{(m_l-1)!}e^{\lambda_l t}\right]'. \tag{5}$$

Then, for any given $t > 0$, the characteristic polynomial of At is $c_{At}(z) = \prod_{i=1}^{l}(z - \lambda_i t)^{m_i}$. By [14, Section 6.1.14,

[3]Although in engineering, it is customary to use real-valued expressions for the modes of A matrix such as $\{e^t \sin(2t), e^t \cos(2t)\}$ in place of $\{e^{(1+2j)t}, e^{(1-2j)t}\}$, development of our results in this section is simplified when the original eigenvalues (possibly complex) are used.

pp. 390], there is a polynomial $r(z) = \sum_{i=0}^{n-1} c_{i+1} z^i$, which satisfies the interpolation conditions: for $j = 0, \cdots, m_i - 1$ and $i = 1, \cdots, l$

$$\frac{d^j r(z)}{dz^j}\Big|_{z=\lambda_i t} = \frac{d^j e^z}{dz^j}\Big|_{z=\lambda_i t} = e^{\lambda_i t}. \tag{6}$$

The coefficients $c_i(t), i = 1 \cdots, n$, depending on t, are uniquely determined by (6). $r(z)$ is said to interpolate e^z and its derivatives at the roots of $c_{At}(z)$.

Let $\alpha_i(t) = c_i(t) t^{i-1}$, $i = 1, \ldots, n$. Then, (6) can be rewritten as

$$\Lambda' \alpha(t) = \xi(t), \tag{7}$$

where the $n \times n$ matrix Λ' depends on λ_i, $i = 1, \ldots, l$ and their multiplicities and is invertible due to the uniqueness of solutions of (6) (see [14, pp. 390]). From the proof of [14, Theorem 6.2.9(a)], one has

$$e^{At} = r(At) = \alpha_1(t) I + \alpha_2(t) A + \cdots + \alpha_n(t) A^{n-1}. \tag{8}$$

It should be pointed out that although determination of $\alpha(t)$ from (6) is unique, there may be other $\alpha(t)$ that satisfies (8) but not (6). In fact, in the proof of Theorem 6.2.9(a) of [14], $r(z)$ is selected from interpolation of e^z and its derivatives at the roots of the minimal polynomial of At. However, here $r(z)$ interpolates e^z and its derivatives at the roots of $c_{At}(z)$, which still leads to $r(At) = e^{At}$ by the same argument as in [14, pp. 387]. This indicates that by choosing different polynomials that annihilate At, one may obtain different expressions of e^{At} in the form of (8). Here we use the particular expression (7), where Λ is invertible.

For any given $x(0)$, $y(t) = Ce^{At} x(0)$ is a linear combination of the modes of A. As a result, it belongs to the class of exponential polynomials: for any $t \in [0, T]$ and $v = [v_1, \cdots, v_n]' \in \mathbb{C}^n$, let

$$g(t) = \sum_{i=1}^{l} \sum_{j=1}^{m_i} v_{i,j} \frac{t^{j-1}}{(j-1)!} e^{\lambda_i t}, \tag{9}$$

where $\lambda_i \in \mathbb{C}, i = 1, \cdots, l$ are the l distinct eigenvalues of A with multiplicity m_i and $m_0 = 0$. Recall that a nonlinear function $g(t)$ is said to be non-trivial if $g(t) \not\equiv 0$. In reference to $\xi(t)$, $g(t) = \xi'(t) v$. Since the elements of $\xi(t)$ are linearly independent, for any $v \neq 0$, g is non-trivial. The following key lemma on the number of zeros of exponential polynomials can be derived from [5, Theorem 3.2.47]. Let μ_T be defined as in (3).

Lemma 1: The number N_T of zeros in $[0, T]$ of a non-trivial exponential polynomial g defined in (9) is bounded by $N_T \leq \mu_T$.

The next lemma is concerned with the rank of M_N in (4).

Lemma 2: If the system in (1) is observable and $N > \mu_T$, then M_N is full rank.

Proof. Suppose A has l distinct eigenvalues λ_i of multiplicity m_i, $i = 1, \ldots, l$. From (8), we have

$$Ce^{At} = [\alpha_1(t), \cdots, \alpha_n(t)] \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = \alpha'(t) W_o. \tag{10}$$

For any N sampling times $t_i \in [0, T], i = 1, 2, \cdots, N$, define

$$\Gamma_N = \begin{bmatrix} \alpha'(t_1) \\ \vdots \\ \alpha'(t_N) \end{bmatrix}. \tag{11}$$

Then,

$$M_N = \Gamma_N W_o. \tag{12}$$

Since W_o is full rank, we only need to show that Γ_N is full rank.

However, by (7) $\Lambda' \alpha(t) = \xi(t)$, which implies

$$\Gamma_N \Lambda = \Xi_N \tag{13}$$

where

$$\Xi_N = \begin{bmatrix} \xi'(t_1) \\ \vdots \\ \xi'(t_N) \end{bmatrix}. \tag{14}$$

Now, by the previous argument, Λ is full rank. Consequently, it remains only to show that Ξ_N is full rank.

For any $\beta \in \mathbb{C}^n$ and $\beta \neq 0$, define $g(t) = \xi'(t)\beta$. Since the elements of $\xi(t)$ are independent, $g(t)$ is non-trivial. $\Xi_N \beta = \begin{bmatrix} g(t_1) \\ \vdots \\ g(t_N) \end{bmatrix} = 0$ means that $g(t)$ has N zeros in $[0, T]$. However, by Lemma 1, the number of zeros of $g(t)$ is bounded by $\mu_T < N$. This contradiction implies that $\beta = 0$. Since β is arbitrary, this proves that Ξ_N is full rank. This completes the proof. □

Proof of Theorem 1. The system output can be written as

$$y(t) = Ce^{At} x(0) + C \int_{t_0}^{t} e^{A(t-\tau)} Bu(\tau) d\tau = Ce^{At} x(0) + \zeta(t),$$

where $\zeta(t)$ is known. For N sampling times $t_i \in [0, T]$ with $i = 1, 2, \cdots, N$, $Y_N = [y(t_i), \cdots, y(t_N)]'$ can be expressed as $Y_N = M_N x(0) + Z_N$, where Z_N is known and M_N is defined in (4). Since the system is observable and $N > \mu_T$, by Lemma 2, M_N is full rank. Consequently, $x(0)$ can be determined uniquely from Y_N. □

IV. Observer Design

We now study observer design when the output measurements are noise corrupted.

A. Observers

For both $t > t_0$ and $t < t_0$, the solution to system (1) can be expressed as $x(t) = e^{A(t-t_0)} x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)} Bu(\tau) d\tau$. Suppose $\{t_i, i = 1, \ldots, N\}$ is a sequence of sampling times. The observer is to estimate $x(t)$, $t_N \leq t < t_{N+1}$ before the next sampling occurs at t_{N+1}.

Since new information about the state is obtained only at the next sampling time t_N, a discrete-time state estimator will first generate an estimate z_N of the state at t_N, which will be used to update the state estimate to $\widehat{x}(t_N) = z_N$, and then the observer will run open loop in (t_N, t_{N+1}).

For $t_i \le t_N$, we have

$$\begin{aligned}&\gamma(t_i)+d_i=y(t_i)\\&=Ce^{A(t_i-t_N)}x(t_N)+C\int_{t_N}^{t_i}e^{A(t_i-\tau)}Bu(\tau)d\tau.\end{aligned}$$

Since the second term is known, it will be denoted by $v(t_i,t_N)=C\int_{t_N}^{t_i}e^{A(t_i-\tau)}Bu(\tau)d\tau$. This leads to the observations

$$Ce^{A(t_i-t_N)}x(t_N)=\gamma(t_i)-v(t_i,t_N)+d_i,\quad i=1,\dots,N. \quad (15)$$

Define

$$\Phi_N=\left[\begin{array}{c}Ce^{A(t_1-t_N)}\\ \vdots\\ Ce^{A(t_{N-1}-t_N)}\\ C\end{array}\right],\ \Gamma_N=\left[\begin{array}{c}\gamma(t_1)\\ \vdots\\ \gamma(t_{N-1})\\ \gamma(t_N)\end{array}\right], \quad (16)$$

$$V_N=\left[\begin{array}{c}v(t_1,t_N)\\ \vdots\\ v(t_{N-1},t_N)\\ 0\end{array}\right],\ D_N=\left[\begin{array}{c}d_1\\ \vdots\\ d_{N-1}\\ d_N\end{array}\right].$$

Then, (15) can be written as

$$\Phi_N x(t_N)=\Gamma_N-V_N+D_N. \quad (17)$$

Suppose that Φ_N is full rank, which will be established in later sections. Then, a least-squares (LS) estimate of $x(t_N)$ is given by

$$z_N=(\Phi_N'\Phi_N)^{-1}\Phi_N'(\Gamma_N-V_N), \quad (18)$$

with estimation error

$$e_N=(\Phi_N'\Phi_N)^{-1}\Phi_N'D_N.$$

The observer will have the state estimate updated at t_N by $\widehat{x}(t_N)=z_N$, and will run as an open loop observer for $t_N\le t<t_{N+1}$, $\dot{\widehat{x}}(t)=A\widehat{x}(t)+Bu(t)$. Due to the state update at t_N, $\widehat{x}(t)$ is discontinuous at t_N, but continuously differentiable in (t_N,t_{N+1}).

B. Recursive Algorithms

z_N can be calculated recursively. Although z_N in (18) appears to be in a LS form, the standard recursive LS algorithm is not applicable, due to the fact that Φ_{N+1} and $\Gamma_{N+1}-V_{N+1}$ are not merely expanded from Φ_N and Γ_N-V_N by adding a new entry.

New information at $N+1$ consists of the sampling time t_{N+1} and the measured value $\gamma(t_{N+1})$, and the following calculated values:

$$M_{N+1}=e^{A(t_{N+1}-t_N)}$$

and

$$w(t_N,t_{N+1})=\int_{t_{N+1}}^{t_N}e^{A(t_N-\tau)}Bu(\tau)d\tau.$$

The relationship among Φ_{N+1}, Γ_{N+1}, V_{N+1} and Φ_N, Γ_N, V_N can be derived as follows:

$$\Phi_{N+1}=\left[\begin{array}{c}\Phi_N e^{A(t_N-t_{N+1})}\\ C\end{array}\right]=\left[\begin{array}{c}\Phi_N M_{N+1}^{-1}\\ C\end{array}\right],$$

$$\Gamma_{N+1}=\left[\begin{array}{c}\Gamma_N\\ \gamma(t_{N+1})\end{array}\right].$$

Moreover,

$$\begin{aligned}&w(t_i,t_{N+1})\\&=\int_{t_{N+1}}^{t_i}e^{A(t_i-\tau)}Bu(\tau)d\tau\\&=\int_{t_N}^{t_i}e^{A(t_i-\tau)}Bu(\tau)d\tau+\int_{t_{N+1}}^{t_N}e^{A(t_i-\tau)}Bu(\tau)d\tau\\&=w(t_i,t_N)+e^{A(t_i-t_N)}\int_{t_{N+1}}^{t_N}e^{A(t_N-\tau)}Bu(\tau)d\tau\\&=w(t_i,t_N)+e^{A(t_i-t_N)}w(t_N,t_{N+1}).\end{aligned}$$

As a result,

$$\begin{aligned}v(t_i,t_{N+1})&=Cw(t_i,t_{N+1})\\&=Cw(t_i,t_N)+Ce^{A(t_i-t_N)}w(t_N,t_{N+1})\\&=v(t_i,t_N)+Ce^{A(t_i-t_N)}w(t_N,t_{N+1})\end{aligned}$$

This implies

$$V_{N+1}=\left[\begin{array}{c}V_N+\Phi_N w(t_N,t_{N+1})\\ 0\end{array}\right].$$

In the following derivation, let

$$P_N=(\Phi_N'\Phi_N)^{-1},\quad K_N=M_N^{-1}P_NC'. \quad (19)$$

Theorem 2: Suppose that for some N_0, Φ_{N_0} is full rank, and let z_{N_0}, P_{N_0}, and K_{N_0} be specified by (18) and (19), respectively. For $N>N_0$, z_N in (18) can be updated recursively by

$$\begin{aligned}z_{N+1}&=M_{N+1}(I-K_{N+1}CM_{N+1})(z_N-w(t_N,t_{N+1}))\\&\quad+P_{N+1}C'\gamma(t_{N+1})\\K_{N+1}&=P_NM_{N+1}'C'(1+CM_{N+1}P_NM_{N+1}'C')^{-1}\\P_{N+1}&=M_{N+1}(I-K_{N+1}CM_{N+1})P_NM_{N+1}'.\end{aligned}$$

Proof. Since $\Phi_{N+1}=\left[\begin{array}{c}\Phi_N M_{N+1}^{-1}\\ C\end{array}\right]$, by the matrix inversion lemma

$$\begin{aligned}&P_{N+1}=(\Phi_{N+1}'\Phi_{N+1})^{-1}\\&=(M_{N+1}^{-1,'}\Phi_N'\Phi_NM_{N+1}^{-1}+C'C)^{-1}\\&=M_{N+1}(P_N^{-1}+M_{N+1}'C'CM_{N+1})^{-1}M_{N+1}'\\&=M_{N+1}(P_N-P_NM_{N+1}'C'(1+CM_{N+1}P_NM_{N+1}'C')^{-1}\\&\quad\times CM_{N+1}P_N)M_{N+1}'.\end{aligned}$$

Let

$$\begin{aligned}K_{N+1}&:=M_{N+1}^{-1}P_{N+1}C'\\&=(P_N-P_NM_{N+1}'C'(1+CM_{N+1}P_NM_{N+1}'C')^{-1}\\&\quad\times CM_{N+1}P_N)M_{N+1}'C'\\&=P_NM_{N+1}'C'(1+CM_{N+1}P_NM_{N+1}'C')^{-1},\end{aligned}$$

which implies

$$P_{N+1}=M_{N+1}(I-K_{N+1}CM_{N+1})P_NM_{N+1}'.$$

Moreover,

$$\begin{aligned}&\Phi_{N+1}'(\Gamma_{N+1}-V_{N+1})\\&=[M_{N+1}^{-1,'}\Phi_N',C']\left(\left[\begin{array}{c}\Gamma_N\\ \gamma(t_{N+1})\end{array}\right]-\left[\begin{array}{c}V_N+\Phi_Nw(t_N,t_{N+1})\\ 0\end{array}\right]\right)\\&=M_{N+1}^{-1,'}\Phi_N'\Gamma_N+C'\gamma(t_{N+1})-M_{N+1}^{-1,'}\Phi_N'V_N\\&\quad-M_{N+1}^{-1,'}\Phi_N'\Phi_Nw(t_N,t_{N+1})\\&=M_{N+1}^{-1,'}\Phi_N'(\Gamma_N-V_N)+C'\gamma(t_{N+1})-M_{N+1}^{-1,'}P_N^{-1}w(t_N,t_{N+1}).\end{aligned}$$

Now,

$$\begin{array}{l} z_{N+1} \\ = P_{N+1}\Phi'_{N+1}(\Gamma_{N+1} - V_{N+1} \\ = M_{N+1}(I - K_{N+1}CM_{N+1})P_N M'_{N+1} M^{-1,'}_{N+1}\Phi'_N(\Gamma_N - V_N) \\ \quad +P_{N+1}C'\gamma(t_{N+1}) - M_{N+1}(I - K_{N+1}CM_{N+1})w(t_N, t_{N+1}) \\ = M_{N+1}(I - K_{N+1}CM_{N+1})z_N + P_{N+1}C'\gamma(t_{N+1}) \\ \quad -M_{N+1}(I - K_{N+1}CM_{N+1})w(t_N, t_{N+1}) \\ = M_{N+1}(I - K_{N+1}CM_{N+1})(z_N - w(t_N, t_{N+1})) \\ \quad +P_{N+1}C'\gamma(t_{N+1}). \end{array}$$

The proof is thus concluded. □

V. Estimation Errors and Persistent Excitation Conditions

A. Estimation Error Representation and Normalization

From (17) and (18), the estimation error for $x(t_N)$ at t_N is

$$\begin{array}{l} e(t_N) = \widehat{x}(t_N) - x(t_N) = (\Phi'_N\Phi_N)^{-1}\Phi'_N D_N \\ = \left(\frac{1}{N^r}\Phi'_N\Phi_N\right)^{-1}\frac{1}{N^r}\Phi'_N D_N \end{array} \quad (20)$$

for any $1/2 < r < 1$.[4] From the expression Φ_N given in (16), we define the (negative) time difference $\tau_i = t_i - t_N$, $0 = \tau_N > \tau_{N-1} = t_{N-1} - t_N > \cdots > \tau_1 = t_1 - t_N$. A typical row of Φ_N is $Ce^{A\tau_i}$. By the Cayley-Hamilton Theorem [22], the matrix exponential can be expressed by a polynomial function of A of order at most $n-1$.

$$e^{At} = \alpha_1(t)I + \cdots + \alpha_n(t)A^{n-1}, \quad (21)$$

where the time functions $\alpha_i(t)$ can be derived by the Lagrange-Hermite interpolation method, see [14], [22] for the algorithms. While the exact calculation of $\alpha_i(t)$ is not important here, it needs to be emphasized that $\alpha_i(t)$ is a linear combination of the modes of A.[5]

This implies

$$Ce^{A\tau_i} = [\alpha_1(\tau_i), \ldots, \alpha_n(\tau_i)]\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = \varphi'(\tau_i)W_o, \quad (22)$$

where $\varphi'(\tau_i) = [\alpha_1(\tau_i), \ldots, \alpha_n(\tau_i)]$ and W_o is the observability matrix. Denote $\Psi_N = \begin{bmatrix} \varphi'(\tau_1) \\ \vdots \\ \varphi'(\tau_N) \end{bmatrix}$. We have

$$\Phi_N = \Psi_N W_o. \quad (23)$$

As a result, for any $r > 0$, $e_N = W_o^{-1}\left(\frac{1}{N^r}\Psi'_N\Psi_N\right)^{-1}\frac{1}{N^r}\Psi'_N D_N$.

Under Assumption 1 (a), W_o^{-1} exists. Convergence results will be established by the following two sufficient conditions: $\frac{1}{N^r}\Psi'_N D_N \to 0$, and $\frac{1}{N^r}\Psi'_N\Psi_N > \beta I$, for some $\beta > 0$. In principle, for the first condition to be true Ψ_N must be bounded. Sampling time sequences must be carefully selected to ensure that the second condition is valid. However, Ψ_N is often unbounded, as shown in the following example.

Example 1: Consider a first-order stable system $\dot{x} = -2x + u$. Then, $n = 1$ and $\alpha_1(t) = e^{-2t}$, and $\alpha_1(\tau_i) = e^{-2(t_i - t_N)}$. For any fixed t_i, when $t_N \to \infty$, $\tau_i \to -\infty$ and $\alpha_1(\tau_i)$ becomes unbounded.

One possible remedy is to normalize the error expressions so that the modified sequence becomes bounded. For this example, one may write $\alpha_1(\tau_i) = e^{2t_N}\widetilde{\alpha}_1(\tau_i)$ with $\widetilde{\alpha}_1(\tau_i) = e^{-2t_i}$. Then, $\Psi_N = e^{2t_N}\widetilde{\Psi}_N$ where $\widetilde{\Psi}_N = e^{-2t_N}\Psi_N$ is bounded, and

$$\left(\frac{1}{N^r}\Psi'_N\Psi_N\right)^{-1}\frac{1}{N^r}\Psi'_N D_N = \left(\frac{e^{2t_N}}{N^r}\widetilde{\Psi}'_N\widetilde{\Psi}_N\right)^{-1}\frac{1}{N^r}\widetilde{\Psi}'_N D_N.$$

This example is generalized to a normalization procedure below. It is noted that only stable modes of A will cause unboundedness of Ψ_N since $\tau_i < 0$. We shall decompose $\varphi(\tau)$ in (22) into two parts $\varphi(\tau) = \varphi_s(\tau) + \varphi_u(\tau)$, where $\varphi_s(\tau)$ contains all modes that are unbounded when $\tau \to -\infty$ and $\varphi_u(\tau)$ contains all modes that are uniformly bounded for $\tau < 0$. More concretely, let the distinct eigenvalues of A be $\Upsilon = \{\nu_i, i = 1, \ldots, m\}$ where ν_i has multiplicity l_i. Decompose Υ into $\Upsilon_s = \{\nu_i \in \Upsilon : \Re(\nu_i) < 0, \text{ or}, \Re(\nu_i) = 0 \text{ and } l_i \geq 2\}$ and $\Upsilon_u = \Upsilon \ominus \Upsilon_s$, where $\Re(z)$ is the real part of z. Then, $\varphi_s(\tau)$ contains all modes corresponding to Υ_s, and $\varphi_u(\tau)$ contains all modes corresponding to Υ_u.

If Υ_s is empty, then $\varphi_s \equiv 0$, Φ_N is bounded, and normalization is not needed. When Υ_s is not empty, define $\nu_{min} = \min_{\nu_i \in \Upsilon_s}\Re(\nu_i)$ and assume that the corresponding eigenvalue has multiplicity l_{min}. Define the normalization factor

$$f(t_N) = \begin{cases} \max\{1, t_N^{l_{min}-1}e^{-\nu_{\min}t_N}\}, & \Upsilon_s \text{ is not empty}, \\ 1, & \Upsilon_s \text{ is empty}. \end{cases}$$

Let $\widetilde{\alpha}_j(t_i, t_N) = \alpha_j(\tau_i)/f(t_N)$. Now, define

$$\widetilde{\Psi}_N = \Psi_N/f(t_N). \quad (24)$$

The elements of $\widetilde{\Psi}_N$ are uniformly bounded. This leads to, for any $r > 0$,

$$\begin{array}{l} e_N = W_o^{-1}\left(\frac{1}{N^r}\Psi'_N\Psi_N\right)^{-1}\frac{1}{N^r}\Psi'_N D_N \\ = W_o^{-1}\left(\frac{f(t_N)}{N^r}\widetilde{\Psi}'_N\widetilde{\Psi}_N\right)^{-1}\frac{1}{N^r}\widetilde{\Psi}'_N D_N. \end{array} \quad (25)$$

Example 2: Suppose a typical $\alpha_j(\tau_i)$ takes the form of $\alpha_j(t) = t^2e^{-2t} + 3te^{-2t} + 7e^{3t}\sin(2t)$. Then $\alpha_j(\tau_i) = (t_i - t_N)^2e^{-2(t_i - t_N)} + 3(t_i - t_N)e^{-2(t_i - t_N)} + 7e^{3(t_i - t_N)}\sin(2(t_i - t_N))$. For $t_N \geq \gamma > 0$ where γ solves $\gamma^2e^{2\gamma} = 1$, $f(t_N) = t_N^2e^{2t_N}$. It follows that

$$\begin{array}{rl} \widetilde{\alpha}_j(t_i, t_N) & = (1 - 2\frac{t_i}{t_N^2} + \frac{t_i^2}{t_N^2})e^{-2t_i} + 3(\frac{t_i}{t_N^2} - \frac{1}{t_N})e^{-2t_i} \\ & +7\frac{e^{-2t_N}}{t_N^2}e^{3(t_i - t_N)}\sin(2(t_i - t_N)) \end{array}$$

which is uniformly bounded since $0 < t_i < t_N$.

This normalization process reduces convergence analysis to the following two sufficient conditions $\frac{1}{N^r}\widetilde{\Psi}'_N D_N \to 0$, and $\frac{1}{N^r}\widetilde{\Psi}'_N\widetilde{\Psi}_N > \beta I$, for some $\beta > 0$.

[4] This expression is valid for any $r > 0$. However, for convergence of $\Phi'_N D_N/N^r$, $r > 1/2$ is usually required, The boundedness of $(\Phi'_N\Phi_N/N^r)^{-1}$ can be established usually for $r < 1$ only.

[5] If A has a real eigenvalue λ of multiplicity m, then the corresponding modes are $e^{\lambda t}$, $te^{\lambda t}$,..., $t^{m-1}e^{\lambda t}$. If A has a complex pair of eigenvalues $\sigma \pm j\omega$ of multiplicity m, then the corresponding modes are $e^{\sigma t}\sin(\omega t)$, $e^{\sigma t}\cos(\omega t)$, $te^{\sigma t}\sin(\omega t)$, $te^{\sigma t}\cos(\omega t)$, ..., $t^{m-1}e^{\sigma t}\sin(\omega t)$, $t^{m-1}e^{\sigma t}\cos(\omega t)$.

A typical element of $\tilde{\Psi}'_N D_N/N^r$ takes the form $\sum_{k=1}^{N} \tilde{\alpha}_j(t_k,t_N)d_k/N^r$. Since $\tilde{\alpha}_j(t_k,t_N)$ is doubly indexed on t_k and t_N, this is a triangularly weighted sum of mixing variables d_k, and classical results on strong convergence of single-indexed weighted sums of mixing variables cannot be applied. The main convergence results will be given by Lemma 3 and Theorem 4 in Section VI.

B. Persistent Excitation Conditions and Sampling Time Sequences

With normalization, $\widetilde{\Psi}_N$ will be bounded and the remaining key issue is the following persistent excitation condition.

PE Condition: For some $1/2 < r < 1$,

$$\beta = \inf_N \sigma_{\min}\left(\frac{f(t_N)}{N^r}\widetilde{\Psi}'_N\widetilde{\Psi}_N\right) > 0, \tag{26}$$

where $\sigma_{\min}(H)$ is the smallest eigenvalue of H for a suitable symmetric matrix H.

In a typical parameter estimation problem, the PE condition is imposed on the input, leading to a condition for identification input design. However, in our problems, $\widetilde{\Psi}_N$ is determined by the matrix A and the sampling sequence only. Hence, the PE condition is a condition on A and the switching time sequence $\{t_i\}$. The PE condition means that (a) $\widetilde{\Psi}'_N\widetilde{\Psi}_N$ is full rank and (b) the smallest eigenvalues of $\widetilde{\Psi}'_N\widetilde{\Psi}_N$ grow at least as fast as $N^r/f(t_N)$. These are non-trivial conditions. They depend on A, C, and the time sequence.

First, (26) is always satisfied under the following conditions. A switching time sequence $\{t_i\}$ in $[0,T]$ is said to be uniformly spread in $[0,T]$ if

$$\lim_{N\to\infty}\frac{1}{N}\Phi'_N\Phi_N = \int_0^T e^{A't}C'Ce^{At}dt. \tag{27}$$

The simplest example is when the switching time sequence is equally spaced in $[0,T]$, for which the limit follows from the Riemann integration. But, there are many other time sequences that also satisfy (27).

Proposition 1: If the system is observable and for a fixed $T>0$, the time sequence is uniformly spread in $[0,T]$, then for sufficiently large N, (26) is always satisfied with $r=1$.

Proof. By hypothesis $\frac{1}{N}(\Phi'_N\Phi_N)$ converges to the integration $\int_0^T e^{A't}C'Ce^{At}dt$, which is full rank for any $T>0$ when (A,C) is observable. Hence, there exists N_0 such that for all $N>N_0$, $\frac{1}{N}\Phi'_N\Phi_N$ is uniformly bounded away from 0. Since W_o is full rank and $f(t_N)$ is bounded in a finite interval, (26) is satisfied. □

More generally, suppose $\{\tau_i\}$ is a time sequence in $[-T,0]$. Since $\alpha_j(\tau)$ are continuous, they are bounded on $[-T,0]$. For $i,j=1,\ldots,n$, suppose $\lambda_{ij} = \lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}\alpha_i(\tau_k)\alpha_j(\tau_k)$ exists. Let M be the $n\times n$ symmetric matrix $M=[\lambda_{ij}]$. Then $\frac{1}{N}\Psi'_N\Psi_N \to M$. As a result, the condition $\inf_N \sigma_{\min}(\frac{1}{N}\Psi'_N\Psi_N) > 0$ is satisfied if $M>0$. On a finite interval, this implies (26).

In general, however, (26) is not always satisfied. When the time sequence is causally and sequentially generated, we show by the following examples that (26) may not hold in some choices of time sequences.

Example 3: Suppose $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $C=[1,0]$. Then (A,C) is observable. If the time sequence is $t_i = i2\pi$, Φ_N contains identical rows of C, which will never be full rank for any N. This example motivates some time sequence selections to avoid such situations.

Example 4: Consider the same system as in Example 3. Suppose that t_i is sequentially generated and approaches 2π. In this case, if we choose the sequence to approach 2π sufficiently fast, the smallest eigenvalues of $\Psi'_N\Psi_N$ become bounded. This implies that $\frac{1}{N^r}\Psi'_N\Psi_N \to 0$, violating (26). This example shows that even in finite horizon scenarios, one needs to choose the time sequence carefully.

We now establish conditions on sampling time sequences that will validate (26).

Case 1: $\varphi_s \not\equiv 0$. In this case, $f(t_N) = t_N^m e^{at_N}$ for some integer m and $a\ge 0$. Let the sampling sequence be selected such that for some $l>0$, $\widetilde{\Psi}'_l\widetilde{\Psi}_l > 0$. Then there exists some $\kappa>0$, for any $N>l$, $\sigma_{\min}(\widetilde{\Psi}'_N\widetilde{\Psi}_N) \ge \kappa > 0$. Consequently, (26) is satisfied if $t_N^m e^{at_N} \ge \frac{\beta N^r}{\kappa}$ which provides a sufficient condition on the sampling time sequences.

Example 5: Returning to Example 1, we have

$$\frac{f(t_N)}{N^r}\Psi'_N\Psi_N = \frac{e^{2t_N}}{N^r}\widetilde{\Psi}'_N\widetilde{\Psi}_N = \frac{e^{2t_N}}{N^r}\sum_{k=1}^{N}e^{-4t_k} \ge e^{-4t_1}\frac{e^{at_N}}{N^r}$$

As a result, by choosing $t_N \ge 2t_1 + \frac{1}{2}\ln(\beta N^r) = 2t_1 + \frac{1}{2}\ln\beta + \frac{r}{2}\ln N$, the PE condition (26) is satisfied.

Case 2: $\varphi_s \equiv 0$. In this case, $f(t_N)=1$ and $\widetilde{\Psi}_N = \Psi_N$. All terms in $\widetilde{\Psi}'_N$ are either bounded or decaying to 0 exponentially, as $t_N \to \infty$. The following example illustrates how the sampling time sequence should be selected.

Example 6: Consider a first-order unstable system $\dot{x} = 3x+u$. Then, $n=1$ and $\alpha_1(t)=e^{3t}$, and $\alpha_1(\tau_i) = e^{3(t_i-t_N)}$. In this case, $\Psi'_N\Psi_N = \sum_{i=1}^{N}e^{-6(t_N-t_i)}$. Letting $t_i = (\ln i)/6$, $\Psi'_N\Psi_N = \sum_{i=1}^{N}e^{-6(t_N-t_i)} = \sum_{i=1}^{N}\frac{i}{N} = \frac{N+1}{2}$, which implies for any $1/2<r<1$, $\Psi'_N\Psi_N/N^r$ will satisfy (26).

We now present a rigorously established sampling time sequence for the PE condition. Suppose that the outputs of the system (1) are sampled at some times $\{t_i\}$ with $t_1=0$ and for some constant $\widetilde{T}>0$

$$t_{i+1}-t_i \ge \log(i+1)-\log i \quad \text{and} \quad t_i \le \log i + \widetilde{T}, \qquad i\ge 1. \tag{28}$$

The following lemma shows that the PE condition is indeed satisfied if the outputs are measured at the sampling times defined in (28).

Theorem 3: Under Assumption 1 (a) and (28), there is a constant $\kappa_1>0$ such that

$$\sigma_{\min}\left(\Psi'_N\Psi_N\right) \ge \frac{\kappa_1 N}{\log^{2[\nu+\mu(\mu-1)]}N}. \tag{29}$$

This theorem implies that under this sampling time sequence, for any $0<r<1$, the PE condition (26) is satisfied. The proof of this theorem, which contains some interesting technical results on function zeros and an improved Lojasiewicz inequality, is highly involved and is postponed to Appendix.

The above examples and theorem indicate a practical guideline in sampling time sequence selections, which will be called *Principle of Logarithmic Time Sequences*: To satisfy the PE condition (26), the sampling time sequence should take the form of $t_i = a + b\ln i$ for some positive constants a and b that depend on the system matrix A.

VI. Convergence Analysis

A. Convergence Analysis in Finite Horizon

In convergence analysis over a finite horizon, we should fix a time interval $[0,T]$. The case of infinite horizon cases will be discussed in the next section. For implementation, time sequences in a finite time interval must be generated causally (no future information is used in selecting the current t_i) and sequentially ($t_{i+1} > t_i$). In this case, since t_i is monotone and bounded, t_i always approaches a limit. As a result, we have $t_i \in [0,T]$ and $t_i \nearrow T$. However, we also allow the case of pre-determined finite N sampling points. In the later case the statement with a limit "$N \to \infty$" should be understood in the following sense: For any desired N, a sampling scheme can be performed such that at least N sampling times occur in this time interval. This problem will become immaterial for infinite horizon cases in which we always assume that the time sequence is generated causally and sequentially.

To proceed, we first establish a rate-of-convergence result regarding the noise. The following result on strong convergence of triangularly weighted sums of ρ^*-mixing variables is essential for this convergence analysis.

Lemma 3: [2, Theorem 2] Let $\{d_k, k \geq 1\}$ be a ρ^*-mixing sequence of identically distributed random variables, $\alpha p > 1, \alpha > 1/2$, and $Ed_1 = 0$ for $\alpha \leq 1$. Suppose that $\{a_{Ni}, 1 \leq i \leq N\}$ is an array of real numbers satisfying

$$\sum_{i=1}^{N} |a_{Ni}|^p = O(N^{\delta}), 0 < \delta < 1. \tag{30}$$

Then $N^{-1/p}\sum_{i=1}^{N} a_{Ni} d_i \to 0$ w.p.1 as $N \to \infty$.

The following theorem is crucial in obtaining the convergence results as well as large-time behavior of the systems.

Theorem 4: Under Assumption 1, for any $r > 1/2$,

$$\frac{1}{N^r}\|\tilde{\Psi}'_N D_N\| \to 0 \quad \text{w.p.1 as} \quad N \to \infty, \tag{31}$$

where $\tilde{\Psi}_N$ is defined by (24).

Proof. To prove the assertion, it suffices to look at the jth component $\sum_{k=1}^{N} \tilde{\alpha}_j(\tau_k) d_k / N^r$ of $\tilde{\Psi}'_N D_N / N^r$. By the transformation in (24), $\tilde{\alpha}_j$ is uniformly bounded, $|\tilde{\alpha}_j| \leq \kappa$, where κ is a generic positive constant.

Select $p = 2$ and $\varepsilon = r - 1/p = r - 1/2$. Since $1/2 < r < 1$, we have $0 < \varepsilon < 1/2$ and $0 < \delta := 1 - 2\varepsilon < 1$. As a result, $\sum_{k=1}^{N}\left(\frac{|\tilde{\alpha}_j(\tau_k)|}{N^{\varepsilon}}\right)^2 \leq \kappa N^{1-2\varepsilon} = \kappa N^{\delta}$. Hence, (30) is satisfied. By Lemma 3,

$$\begin{aligned}\frac{1}{N^{1/2}}\sum_{k=1}^{N}\frac{\tilde{\alpha}_j(\tau_k)}{N^{\varepsilon}} d_k &= \frac{1}{N^{1/2+\varepsilon}}\sum_{k=1}^{N}\tilde{\alpha}_j(\tau_k) d_k \\ &= \frac{1}{N^r}\sum_{k=1}^{N}\tilde{\alpha}_j(\tau_k) d_k \to 0 \text{ w.p.1 as } \quad N \to \infty.\end{aligned}$$

□

Theorem 5: Under Assumption 1 and (26), the following assertions hold:

(a) $e(t_N) \to 0$ w.p.1 as $N \to \infty$.

(b) If in addition, $\sum_{j \geq i} |Ed_i d_j| < \infty$, then $\zeta = \sup_N N^{2r-1} Ee'(t_N) e(t_N) < \infty$.

Proof. Note that assertion (a) follows immediately from Theorem 4.

To prove (b), since the system is observable, W_o is invertible. It follows from (23) and (25) that

$$e_N = \frac{1}{f(t_N)} W_o^{-1} \left(\frac{1}{N^r}\widetilde{\Psi}'_N \widetilde{\Psi}_N\right)^{-1} \frac{1}{N^r}\widetilde{\Psi}'_N D_N.$$

Now,

$$\frac{1}{N^r}\widetilde{\Psi}'_N D_N = \begin{bmatrix} \frac{1}{N^r}\sum_{i=1}^{N}\widetilde{\alpha}_1(\tau_i) d_i \\ \vdots \\ \frac{1}{N^r}\sum_{i=1}^{N}\widetilde{\alpha}_n(\tau_i) d_i \end{bmatrix}. \tag{32}$$

Under the hypothesis, since all $\widetilde{\alpha}_i(\tau)$ are continuous, uniformly bounded, and τ_i are uniformly bounded in $[-T, 0]$, $\forall i, j, |\widetilde{\alpha}_j(\tau_i)| \leq \kappa$ for some generic constant $\kappa > 0$. It follows that for a typical term $\frac{1}{N^r}\sum_{i=1}^{N}\widetilde{\alpha}_j(\tau_i) d_i$ in (32),

$$\begin{aligned} &E\Big(\frac{1}{N^r}\sum_{i=1}^{N}\widetilde{\alpha}_j(\tau_i) d_i\Big)^2 \\ &= \frac{1}{N^{2r}}\sum_{i=1}^{N}\sum_{j=1}^{N}\widetilde{\alpha}_j(\tau_i)\widetilde{\alpha}_j(\tau_j) E d_i d_j \\ &\leq \frac{\kappa}{N^{2r}}\sum_{i=1}^{N}\sum_{j\geq i}^{N}|Ed_i d_j| \leq \frac{\kappa}{N^{2r-1}}. \end{aligned}$$

This implies

$$N^{2r-1}E\Big(\frac{1}{N^r}\sum_{i=1}^{N}\widetilde{\alpha}_j(t_i) d_i\Big)^2 \leq \kappa. \tag{33}$$

Moreover, by (26), $(\frac{1}{N^r}\widetilde{\Psi}'_N\widetilde{\Psi}_N)^{-1}$ exists and $\sigma_{\max}\left((\frac{1}{N^r}\widetilde{\Psi}'_N\widetilde{\Psi}_N)^{-1}\right) \leq \frac{1}{\beta}$, where $\sigma_{\max}(\cdot)$ is the largest eigenvalue. This, together with the existence of W_o^{-1} and (33) implies

$$N^{2r-1} E e'_N e_N \leq \kappa < \infty, \tag{34}$$

for some constant κ □

Under some mild conditions, convergence of $\widehat{x}(t_N)$ to $x(t_N)$ implies that $\widehat{x}(t) - x(t) \to 0$ for any $t_N \leq t < t_{N+1}$. Let $T_h > 0$ be a finite time horizon and $e(t) = \widehat{x}(t) - x(t)$. Then it is easily seen that $e(t_N) = e_N$.

Theorem 6: Under Assumption 1 and (26), for $t \in [t_N, t_N + T_h]$ and $t_N + T_h < t_{N+1}$ $\max_{t_N \leq t \leq t_N + T_h} Ee'(t)e(t) \to 0, \quad N \to \infty$.

Proof. This follows from the fact that for $t \in [t_N, t_N + T_h]$, the estimator runs open loop and $e(t) = e^{A(t-t_N)} e(t_N)$. Since T_h is finite and $e^{A(t-t_N)}$ is continuous, there exists a constant $\kappa > 0$ such that $\max_{t_N \leq t \leq t_N + T_h} \sigma_{\max}(e^{A(t-t_N)}) = \kappa < \infty$. Consequently, for any $t \in [t_N, t_N + T_h]$ $Ee'(t)e(t) \leq \kappa Ee'(t_N)e(t_N) \to 0$ as $N \to \infty$. □

B. Convergence Analysis over Infinite Horizons

The convergence properties of the previous section are established over a finite time interval. For stability analysis of the closed-loop system, infinite time horizon must be considered. We now investigate observation errors under certain unbounded switching time sequences. In this section, time sequences will always be causally and sequentially generated.

1) Relationship between e_N and $e(t)$: The observer error sequence $e(t_N)$ is said to be in l^2 if $\sum_{N=1}^{\infty} Ee'(t_N)e(t_N) < \infty$ and $e \in L^2$ (mean square integrable) if $\int_0^\infty Ee'(t)e(t)dt < \infty$.

Let $|t_{N+1} - t_N| \leq q(N)$. The next lemma claims that for dealing with convergence properties of $e(t)$ we may concentrate on the sequence e_N.

Lemma 4: If $q(N)$ is uniformly bounded and $t_N \to \infty$, then (a) $e_N \to 0$ w.p.1 as $N \to \infty$ implies that $e(t) \to 0$ w.p.1 as $t \to \infty$. (b) $e_N \to 0$ in mean square as $N \to \infty$ implies that $e(t) \to 0$ in mean square as $t \to \infty$. (c) $\sqrt{q(N)}e_N \in l^2$ implies $e \in L^2$.

Proof.

(a) Since the observer is running open loop for $t \in [t_N, t_{N+1})$, we have $e(t) = e^{A(t-t_N)}e(t_N)$. Now if $t_{N+1} - t_N$ is uniformly bounded, there exists a constant $\kappa > 0$ such that

$$\max_{t_N \leq t \leq t_{N+1}} e'(t)e(t) \leq \kappa e'(t_N)e(t_N). \tag{35}$$

This and the hypothesis $t_N \to \infty$ implies that $e(t_N) \to 0$ as $N \to \infty$ w.p.1 leading to $e(t) \to 0$ w.p.1. as $t \to \infty$.

(b) The inequality (35) implies further that $\max_{t_N \leq t \leq t_{N+1}} Ee'(t)e(t) \leq \kappa Ee'(t_N)e(t_N)$. Hence, $Ee'(t_N)e(t_N) \to 0$, as $N \to \infty$ implies that $Ee'(t)e(t) \to 0$ as $t \to \infty$.

(c) Moreover,

$$\begin{aligned}
&\int_0^\infty Ee'(t)e(t)dt \\
&= \lim_{N\to\infty} \int_0^{t_N} Ee'(t)e(t)dt \\
&\leq \kappa \lim_{N\to\infty} \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} Ee'(t_{j-1})e(t_{j-1})dt \leq \kappa \sum_{j=1}^{\infty} q(j)Ee_j'e_j.
\end{aligned}$$

As a result, $\sqrt{q(N)}e(t_N) \in l^2$ implies that $e \in L^2$. □

2) MS and Strong Convergence:

Case 1: $\varphi_s(\tau) \neq 0$

In this case, $\nu_{\min} \leq 0$ and hence $f_s(t_N) \to \infty$, $t_N \to \infty$ or $f_s(t_N) = 1$ (if $\nu_{\min} = 0$ and $l = 1$) as $t_N \to \infty$, $\tau_i = t_i - t_N \to -\infty$. Since $1/f(t_N)$ is bounded, the next theorem follows from Theorem 5.

Theorem 7: Under Assumption 1 and (26), if $\varphi_s(\tau) \neq 0$, then $\zeta = \sup_N N^{2r-1}Ee'(t_N)e(t_N) < \infty$ and $e(t_N) \to 0$ w.p.1 as $N \to \infty$.

Case 2: $\varphi_s(\tau) = 0$

In this case, $\varphi(\tau) = \varphi_u(\tau)$ with $|\varphi_u(\tau)| \to 0, \quad \tau \to -\infty$. As a result, when $t_N \to \infty$, for any fixed i, $\varphi_j(\tau_i) = \varphi_j(t_i - t_N) \to 0$, rendering $\frac{1}{N^r}\Psi_N'\Psi_N \to 0$ and the PE condition is lost.

To resolve this issue, we extract the dominant mode as in Case 1. Since $f(t_N) \to 0$ in this case, the unstable modes may result in divergence. We now resort to generation of switching time sequences to restore convergence.

Assumption 2: For some $\nu > \nu_{\min} > 0$, the switching-time sequence $\{t_N\}$ satisfies (a) $t_N \to \infty$ as $N \to \infty$. (b) For some $0 < r_0 < 1/2$, $e^{2\nu t_N}/N^{r_0} \to 0$ as $N \to \infty$. (c) $|t_{N+1} - t_N| \leq q(N)$ such that for any $r_1 > 0$

$$\sum_{N=1}^{\infty} \frac{q(N)}{N^{r_1}} < \infty. \tag{36}$$

Remark 3: We show now that a sequence $\{t_N\}$ satisfying Assumption 2 for any chosen r_0 can be constructed. This remark also provides a typical choice of such a $\{t_N\}$ sequence. Choose t_N from

$$\frac{e^{2\nu t_N}}{N^{r_0}} = \frac{1}{N^\varepsilon}, \quad 0 < r_0 < 1/2, \quad 0 < \varepsilon < \min\{r_0, 1/2 - r_0\}. \tag{37}$$

Then $t_N = \frac{r_0 - \varepsilon}{2\nu} \ln N$. Clearly, $t_N \to \infty$ and $e^{2\nu t_N}/N^{r_0} \to 0$. Now,

$$\Delta t_N = t_{N+1} - t_N = \frac{r_0 - \varepsilon}{2\nu}(\ln(N+1) - \ln N).$$

For large N, $|\Delta t_N| \leq q(N) = O(1/N)$ due to

$$\lim_{N\to\infty} \frac{\frac{r_0-\varepsilon}{2\nu}(\ln(N+1) - \ln N)}{1/N} = \frac{r_0 - \varepsilon}{2\nu}.$$

It follows that $q(N)/N^{r_1} = O(1/N^{1+r_1})$, which implies $\sum_{N=1}^{\infty} q(N)/N^{r_1} < \infty$.

Remark 4: Although the generated sampling time sequences are unbounded (i.e., $t_N \to \infty$), the sampling intervals $(t_N - t_{N-1})$ become progressively smaller. For practical implementation, when the switching sensor has accuracy limit, time delay, sensor noise, there is a limitation on the sampling interval. We can easily accommodate such a limitation by stopping reduction of sampling intervals when they fall below precision levels of the sensors or the control actuators. This stopping rule will result in a small residual and irreducible error. This is not a new phenomenon. For any convergence or stability analysis, measurement precision will impose a limitation. Consider an open-loop system $\dot{x} = 2x + u$, the feedback control $u = -4x$ achieves a closed-loop system $\dot{x} = -2x$ which is exponentially stable. However, if measurement or actuator implementation has limited accuracy, the true feedback will become $u = -4x + d$ with d a bounded error from measurement. Then the closed-loop system becomes $\dot{x} = -2x + d$. x will converge exponentially to a bounded set, but not to 0. Our theoretical development will always be subject to such implementation constraints which will not be discussed further.

Theorem 8: Under Assumption 1, (26), and 2, with r_0 satisfying $0 < r_0 < \min\{2r - 1, 1/2\}$, (a) $e(t) \to 0$ in the MS sense. (b) $e \in L^2$.

Proof. From the definition of $\nu > \nu_{\min}$, since $\widetilde{\alpha}_j(\tau_i)$ is bounded, we have $|\tilde{\alpha}_j(\tau_i)/f_u(t_N)| \leq \kappa e^{\nu t_N}$ for some constant κ.

(a) Following the derivation steps in (34), we arrive at $N^{2r-1}Ee'(t_N)e(t_N) \leq \kappa e^{2\nu t_N}$ for some constant κ. This leads to

$$Ee'(t_N)e(t_N) \leq \kappa \frac{e^{2\nu t_N}}{N^{2r-1}} \leq \kappa \frac{e^{2\nu t_N}}{N^{r_0}} \to 0 \tag{38}$$

by Assumption 2. Now, Lemma 4 implies $e(t) \to 0$ in the MS sense.

(b) By (38) and Assumption 2

$$Ee'_N e_N \le \kappa \frac{e^{2\nu t_N}}{N^{2r-1}} = \kappa \frac{e^{2\nu t_N}}{N^{r_0} N^{r_1}} \le \kappa \frac{1}{N^{r_1}}$$

with $r_1 = 2r - 1 - r_0 > 0$, which implies $q(N)Ee'_N e_N \le \kappa q(N)/N^{r_1}$. By Assumption 2, $\sum_{N=1}^{\infty} q(N)/N^{r_1} < \infty$, namely $\sqrt{q(N)}e_N \in l^2$. By Lemma 4, $e \in L^2$. □

3) MS Exponential Convergence: We now present conditions under which exponential convergence rates of $e(t)$ can be obtained. We show that $e(t_N)$ converges to 0 exponentially in the MS sense and w.p.1.

Case 1: $\varphi_s(\tau) \neq 0$

Theorem 9: Under the assumptions of Theorem 7, if the switching time sequence is $t_N = \gamma \ln N$ for some $\gamma > 0$, then for any $0 < \eta < (2r-1)/\gamma$ $e^{\eta t_N} Ee'(t_N)e(t_N) \to 0$ as $N \to \infty$.

Proof. From $t_N = \gamma \ln N$ we obtain $N = e^{t_N/\gamma}$. By Theorem 7, $\zeta = \sup_N N^{2r-1} Ee'(t_N)e(t_N) < \infty$, which implies $e^{t_N(2r-1)/\gamma} Ee'(t_N)e(t_N) \le \kappa < \infty$. Since $0 < \eta < (2r-1)/\gamma$ and $t_N \to \infty$,

$$e^{\eta t_N} Ee'(t_N)e(t_N) \le e^{t_N(\eta-(2r-1)/\gamma)} \kappa \to 0 \quad \text{as} \quad N \to \infty.$$

□

Case 2: $\varphi_s(\tau) = 0$

Theorem 10: Under the assumptions of Theorem 8 and suppose t_N is generated from (37), if for some $r > 1/2$ $\beta = \inf_N \sigma_{\min} \widetilde{\Psi}'_N \widetilde{\Psi}_N / N^r > 0$, then there exists $\eta > 0$ such that $e^{\eta t_N} Ee'(t_N)e(t_N) \to 0$ as $N \to \infty$.

Proof. By (38) and (37)

$$Ee'(t_N)e(t_N) \le \kappa \frac{e^{2\nu t_N}}{N^{2r-1}} \le \kappa \frac{e^{2\nu t_N}}{N^{r_0}} = \kappa \frac{1}{N^{\varepsilon}} = \kappa e^{-\frac{2\nu\varepsilon}{r_0-\varepsilon} t_N}.$$

Let $0 < \eta < \frac{2\nu\varepsilon}{r_0-\varepsilon}$. Then $e^{\eta t_N} Ee'(t_N)e(t_N) \le \kappa e^{(-\frac{2\nu\varepsilon}{r_0-\varepsilon}+\eta)t_N}$. This proves $e^{\eta t_N} Ee'(t_N)e(t_N) \to 0$ as $N \to \infty$. □

4) Almost Sure Exponential Convergence: Here, we derive exponential convergence rates in the almost sure sense for $e(t_N)$.

By a certain choice of switching time sequences, Theorem 4 implies exponential convergence in the almost sure sense. Suppose that the switching time sequence is $t_N = \gamma \ln N$ which implies $e^{t_N} = N^{\gamma}$.

Theorem 11: Under the Assumption 1 and (26), if $\varphi_s(\tau) \neq 0$, then there exists some $\eta > 0$ such that

(i) $e^{\eta t_N} \|e(t_N)\| = O(1)$ w.p.1 as $N \to \infty$.

(ii) $e^{\eta t} \|e(t)\| = O(1)$ w.p.1 as $t \to \infty$.

Proof. To prove (i), note that by (25) and under (26), for some $r_1 > r_0$

$$\|e^{\eta t_N} e(t_N)\| \le \kappa \frac{N^{\eta\gamma}}{N^{r_1}} \|\widetilde{\Psi}'_N D_N\| \le \frac{\kappa}{N^{r_1-\eta\gamma}} \|\tilde{\Psi}'_N D_n\|.$$

Choose $r_1 - \eta\gamma \ge r_0$ that is given in Theorem 4, Theorem 4 immediately yields the claim.

To prove (ii), since $t_N \to \infty$ as $N \to \infty$, for any $t > 0$, there exists N such that $t_N \le t < t_{N+1}$. Also, $|t - t_N| \le |t_{N+1} - t_N| = |\ln(N+1) - \ln N| \le 1$. Since $e(t) = e^{A(t-t_N)} e(t_N)$, $t_N \le t < t_{N+1}$ we have $e^{\eta t} \|e(t)\| \le \kappa e^{\eta t} \|e(t_N)\| \le \kappa e^{\eta(t-t_N)} \le \kappa$. □

The case $\varphi_s(\tau) = 0$ can be directly obtained from Theorem 4 by a similar treatment of Theorem 11 for any switching time sequence with $|t_i - t_{i-1}|$ uniformly bounded, where $i \ge 1$.

Theorem 12: Under the Assumption 1 and (26), if $\varphi_s(\tau) = 0$, there exists some $\eta > 0$ for which $e^{\eta t_N} \|e(t_N)\| = O(1)$ w.p.1 as $N \to \infty$ and $e^{\eta t} \|e(t)\| = O(1)$ w.p.1 as $t \to \infty$.

VII. Generation of Switching Time Sequences

Estimation of the state utilizes the information from the sampling-time/ switching-time sequence $\{t_i\}$ and the corresponding values $\gamma(t_i)$. Consequently, it is essential to generate such a sequence. In practical systems, the irregular sampling sequences can be generated via different means, depending on system settings. We list some of these sampling-time generating mechanisms for which the results of this paper may be applied.

A. Directly Controlled Sampling

Multi-rate sampled systems are a classical topic. Such sampling schemes implicitly assume that sampling times can be designed. We refer the reader to some standard textbooks and monographs [4], [7], [24] for sampled systems with uniform or non-uniform sampling intervals. Classical treatment of this topic is mostly limited to multi-rate sampling and does not seek active utility of such a capability in state estimation. Advanced analog-to-digital conversion units (A-D) allow selection of sampling times beyond periodic sampling schemes. In these cases, the sampling times can be directly controlled. The results of this paper provide some insights and methodologies on how such advanced capabilities might be utilized to achieve fast convergence of state estimation.

B. Event-Triggered Sampling and Signal Quantization

Departing from traditional sampling mechanisms, sampling times may be event triggered. Typically, if a binary-valued or quantized sensor is used in signal measurement, a sampled value is generated only when the output of such sensors switches values. Each switching of the sensor output represents an event that provides a sampled value of the system output at the switching time. Event-based sampling usually involves fixed thresholds, open-loop stable systems or just a class of bounded signals [3], [23]. For example, photoelectric sensors for positions of robot movements, Hall-effect sensors for speed and acceleration in electrical machines, EGO oxygen sensors in automotive emission control will provide meaningful values of the system outputs only at the time of sensor switching. In a distributed sensor network, a grid of location sensors are allocated in an area, creating a network of checkpoints. When a mobile agent, such as a vehicle, a robot, a soldier, or an airplane, passes through a sensor checkpoint, the event will generate a position signal at that time instant. The set of the time instants forms an irregular sampling-time sequence.

C. Binary-Valued Sensors with Input Control

Suppose that the output of the system (1) is measured by a sensor whose output $s(t) \in \mathbb{R}$ is binary-valued with a fixed threshold γ. The sensor can be represented by

$$s(t) = S(y(t)) = \begin{cases} 1, & \text{if } y(t) \le \gamma; \\ 0, & \text{if } y(t) > \gamma. \end{cases} \tag{39}$$

At a switching time t_i, the sensor threshold value γ is an approximation of $y(t_i)$ in that $\gamma = y(t_i) - d_i$, which will be used to estimate the state. Since a binary sensor provides information about $y(t)$ only when it switches, one may actively control the input to cause $y(t)$ to pass the sensor threshold γ.

In [35], sampling times are generated by controlling the input so that the sensor output will switch values within a designated time interval, under assumption that the system is state controllable.

Assumption 3: 1) The prior information on the initial state $x(0)$ is given by a compact set Ω_0, namely, $x(0) \in \Omega_0$.
2) System (1) is completely state controllable and completely state observable.

Theorem 13: [35] Under Assumption 3, for any $t_1 > 0$, there exists a causal control input $u(t)$ for $t \in [0, t_1]$ such that $s(t)$ switches at least once in $(0, t_1)$.

Corollary 1: [35] Under Assumption 3, for any $t_1 > 0$ and for any given integer m, there exists a causal control input $u(t)$, $t \in (0, t_1)$ such that $s(t)$ switches at least m times in $(0, t_1)$.

The above results indicate that by actively controlling the input, one can always generate a desired switching-time sequence.

D. Binary-Valued Sensors with Threshold Control

Another approach of controlled generation of sampling-time sequences is to control the sensor threshold. Suppose that the output of the system (1) is measured by a sensor whose output $s(t) \in \mathbb{R}$ is binary-valued with a possibly time-varying threshold $\gamma(t)$. The sensor can be represented by

$$s(t) = S(y(t)) = \begin{cases} 1, & \text{if } y(t) \le \gamma(t); \\ 0, & \text{if } y(t) > \gamma(t). \end{cases} \tag{40}$$

One possible scheme of threshold control is illustrated in control problem for a power system shown in Figure 1. The voltage $v(t)$ is the output of a system whose states are to be estimated. The reference voltage $v_R(t)$ is generated by a circuit whose value can be adjusted. This can be easily realized by operational amplifiers, DC-DC converters, etc. The binary sensor is a simple sign detector. When the sign detector switches its value at t_i, it triggers the reference value $v_R(t_i)$ to be coarsely quantized to $\widehat{v}_R(t_i)$ and hence is subject to errors, and sent over the noise-corrupted network to the receiver. The receiver receives $w(t_i) = v(t_i) + d_i$ as a noise corrupted observation on $v(t_i)$ at the switching time, and has a clock so the time t_i is also recorded. The switching time t_i and the received value $w(t_i)$ constitute the information for observer design and feedback.

Let $\|\cdot\|$ be a vector norm.

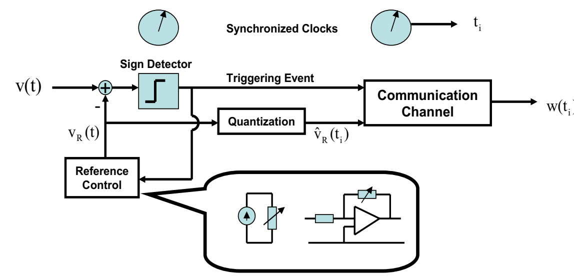

Fig. 1. Controlled threshold and networking example

Assumption 4: At $t = 0$, the information on the state $x(0)$ is given by a bounded set Ω.

Theorem 14: Under Assumption 4, given a switching time t_i, for any $T > t_i$, there exists a continuous threshold function $\gamma(t)$ for $t \in [t_i, T]$ such that $s(t)$ switches at least once in (t_i, T).

Proof. Consider first the case $y(t_i) > \gamma(t_i)$, that is, $s(t_i) = 0$. We will show that there exists a function $\gamma(t)$ for $t \in (t_i, T)$ such that $y(T) < \gamma(T)$. By continuity, this implies that $s(t)$ switches from 0 to 1 at least once at $t_{i+1} \in (t_i, T)$.

For $t > 0$, $y(t) = Ce^{At}x(0) + v(t, 0)$. Under Assumption 4, there exist known constants $\alpha > 0$ and $\beta \ge 0$ ($\beta = 0$ if the system is stable) such that

$$y(t) \le \alpha e^{\beta t} + v(t, 0), \quad t > 0. \tag{41}$$

Let the threshold function be selected as

$$\gamma(t) = \gamma(t_i) + \alpha e^{\eta_i (t - t_i)} + v(t, 0) \tag{42}$$

with some $\eta_i > \beta$ which is to be specified later. Since $v(t, 0)$ is continuous, $\gamma(t)$ is continuous. To guarantee $y(T) < \gamma(T)$, we only need to take sufficiently large η_i such that

$$T > \frac{\eta_i t_i}{\eta_i - \beta} + \frac{|\gamma(t_i)|}{\alpha(\eta_i - \beta)}, \tag{43}$$

which gives

$$e^{\eta_i (T - t_i)} - e^{\beta T} \ge \eta_i (T - t_i) - \beta t = \frac{|\gamma(t_i)|}{\alpha}. \tag{44}$$

Then, from (41) and (42), we have $y(T) < \gamma(T)$. Consequently, $y(t_{i+1}) = \gamma(t_{i+1})$ will occur at some $t_{i+1} \in (t_i, T)$.

The case of $y(t_i) < \gamma(t_i)$ can be similarly achieved by (44) if we let $\gamma(t) = \gamma(t_i) - \alpha e^{\eta_i (t - t_i)} + v(t, 0)$, where $\eta_i > \beta$ satisfies (43), since $y(t) \ge -\alpha e^{\beta t} + v(t, 0)$, $t > 0$. □

Theorem 14 can be applied repeatedly to reach the following conclusion.

Corollary 2: Under Assumption 4, for any given time interval and for any given integer m, there exists a threshold design $\gamma(t)$, such that $s(t)$ switches at least m times in the time interval.

We should point out that for practical algorithm implementations, we may use different $\gamma(t)$, as long as the sensor switches frequently to provide information for state estimation.

Example 7: Consider a system with $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -3 & -3 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}$, $C = [1\ 0\ 0\ 0]$.

The input to the system is a step input $u(t) = 0$, $t < 0$ and $u(t) = 10$, $t \geq 0$, and initial state is $x(0) = [10\ 0\ 0\ 0]'$. This implies that the initial output is $y(0) = 10$. Initial threshold value is $\gamma(0) = 5$. For $t > 0$, the threshold $\gamma(t)$ is controlled by increasing its value when $s(t) = 0$ and decreasing its value when $s(t) = 1$, leading to two functions of $\gamma(t)$. In this simulation, both functions are linear, although exponential functions can be used also. Simulation results are shown in Figure 2.

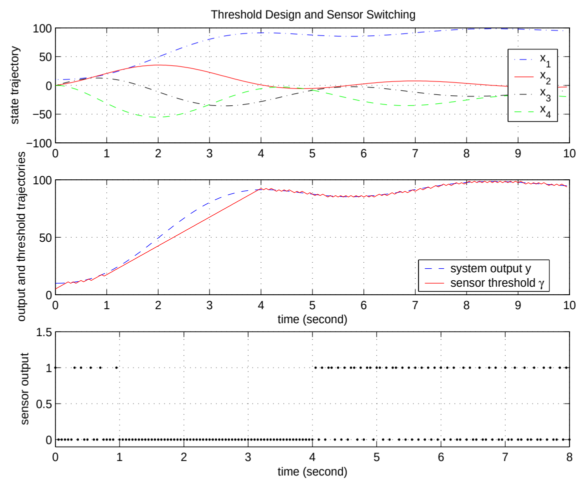

Fig. 2. Threshold design and sensor switching

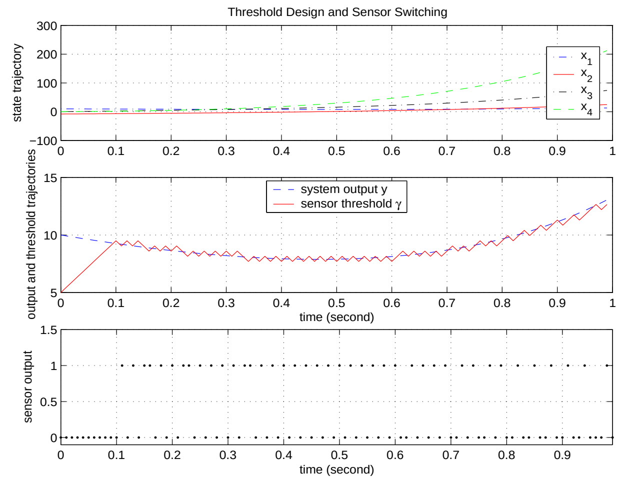

Fig. 3. Signal trajectories

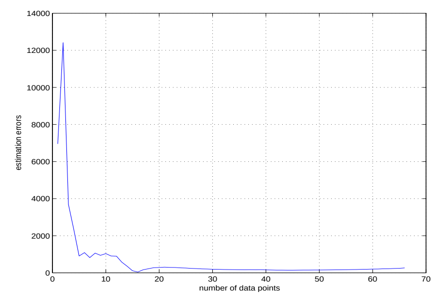

Fig. 4. State estimation errors

VIII. An Illustrative Example

We provide an example to demonstrate how the methods of this paper may be used. In the example, the time-varying sensor threshold is compared to the noise-corrupted plant output by using a binary comparisor, representing the binary sensor. When its output switches values, the threshold value at that time instant is used as the measured value of the output.

Example 8: Consider now an unstable system $\dot{x} = Ax + Bu, y = Cx$ with $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -1 & 5 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}$, $C = [1, 0, 0, 0]$ The unknown initial state is $x(0) = [10, -8, 0, 0]'$. We first run the system open-loop for a short period of time with a constant control input $u = 10$. The threshold is controlled to increase or decrease on the basis of sensor output switching, using the threshold control algorithms. To take into consideration of the threshold control and time measurement constraints, we impose a bound on the maximum slope of threshold changes at 45 per second. Simulation is performed and the collected data on the binary sequences are used to estimate $x(t)$. Figure 3 depicts the signal trajectories, including the states, the output, the threshold control, and the sensor output. Figure 4 shows state estimation errors as a function of data points.

IX. Conclusions

This paper has resolved the observability issue: If the density of the sampling times exceeds a critical threshold, the sampled system is always observable, independent of the actual times. This finding provides a foundation for state estimation, output feedback, and other control activities that involve irregular sampling due to, for example, communication channel interruptions, event-triggered sampling, or other system constraints.

This paper also introduces a method of designing observers for linear time invariant systems with irregular sampling-time sequences, which can be generated either passively in event-triggered sampling or by actively controlling the system input or sensor threshold when the sensor is binary-valued. By selecting the sampling-time/switching-time appropriately, the persistent excitation condition is validated and convergent state observers can be developed. It is shown that convergence in mean square, with probability one, as well as almost sure exponential rates can be guaranteed when the switching time sequence is suitably generated. The results of this paper can be used in feedback design for stability and performance on the basis of state estimation.

X. Appendix: Proof of Theorem 3

The proof of Theorem 3 relies on several technical results, which are of interests by themselves and will be presented first.

A. Improved Lojasiewicz Inequality

The smallest eigenvalue of $\Psi_N' \Psi_N$ relies on some lower bounds on modulus of exponential polynomials in the form of (9), where $f(t)$ can be viewed as a confinement of a complex exponential polynomial on the real axis. The global Lojasiewicz inequality (see [5, p. 210]) provides a lower bound on the module of an analytic function. It will be extended here to estimate the lower bound in (29). In this subsection, we will derive a key inequality which is essential to estimate the smallest eigenvalue of matrix. First, we need some preliminary knowledge on exponential polynomial.

Definition 2: An exponential polynomial of $z \in \mathbb{C}$ (the set of complex numbers) is an entire function $F(z)$ of the form

$$F(z) = \sum_{1 \le j \le l} p_j(z) e^{\mu_j z}, \tag{45}$$

where $\mu_j \in \mathbb{C}, j = 1, \ldots, l$ are distinct, and $p_j(z) = \sum_{k=0}^{n_j} p_{j,k} z^k$, $p_{j,k} \neq 0$ are nonzero polynomials of degree n_j.

The degree of $F(z)$ is

$$d^o F := \sum_{j=1}^{l} n_j + l - 1, \tag{46}$$

and the norm of F is

$$\|F\| := \sum_{j=1}^{l} \sum_{k=0}^{n_j} |p_{j,k}|. \tag{47}$$

Now, $g(t)$ in (9) is the limitation of

$$g(z) := \sum_{j=1}^{l} \sum_{k=1}^{m_j} \beta_{j,k} z^{k-1} e^{\lambda_j z} \tag{48}$$

to $z = t$, where $\lambda_j, j = 1, \cdots, l$ are the l distinct eigenvalues of A (also called frequencies of $g(z)$ in complex analysis) with multiplicity m_j, respectively. Note that $\sum_{j=1}^{k} m_j = n$. The coefficients $\beta_{j,k} \in \mathbb{C}$ and here

$$p_j(z) = \sum_{k=1}^{m_j} \beta_{j,k} z^{k-1} = v_{j,k} \frac{z^{k-1}}{(k-1)!}, \tag{49}$$

where $v = (v_1, \cdots, v_n) \in \mathbb{C}^n$ with $\|v\| = 1$. Obviously, $p_j(z) \neq 0$.

According to Definition 2 and (46), the degree of $g(z)$ depends on the coefficients $\beta = \{\beta_{j,k}\}$. If $\forall j, \beta_{j,m_j} \neq 0$, then

$$d^o g = \sum_{j=1}^{l} (m_j - 1) + l - 1 = n - 1. \tag{50}$$

In general, if the leading coefficient (non-zero $\beta_{j,k}$ with the highest index k) of $p_j(z)$ is β_{j,k_j}, then $d^o g = \sum_{j=1}^{l} (k_j - 1) + l - 1 \le n - 1$. We denote the leading coefficient by

$$a_j(g) = \beta_{j,k_j}. \tag{51}$$

By (47), the norm of g is

$$\|g\| = \sum_{j=1}^{l} \sum_{k=1}^{m_j} |\beta_{j,k}| \ge \frac{1}{\mu!}, \tag{52}$$

where $\mu = \max_{1 \le j \le l} m_j - 1$. Furthermore, define the supporting function by $H(z) := \max_j \Re(\lambda_j z)$, where we use $\Re(z)$ and $\Im(z)$ denote the real part and the imaginary parts of $z \in \mathbb{C}$, respectively.

Now, we proceed to estimate the modulus of g. Let $V = \{z \in \mathbb{C} : g(z) = 0\} \neq \emptyset$ be the set of zeros of $g(z)$ and $d(z, V)$ denotes the distance between z and V. The global Lojasiewicz inequality (see [5, p. 210]) gives a lower bound on $|g(z)|$ by

$$|g(z)| \ge \kappa \left(\min_{1 \le j \le l} |a_j(g)| \right) \frac{d(z, V)^\nu}{(1 + |z|)^\mu} e^{H(z)},$$

where $\nu = 9 d^o g$, $\widetilde{\mu} = 8 \max_{1 \le j \le l} (m_j - 1)$, and $\kappa > 0$ can be explicitly estimated in terms of λ_j and m_j of p_j. However, the leading terms $|a_j(g)|$ may be very small even if the norm of g is a given value. So, we make some modifications to the global Lojasiewicz inequality to estimate the modulus of g by $\|g\|$. It will be useful in dealing with the eigenvalues of Hermite matrices in the subsequent arguments.

The following modification is based on the argument of Grudzinski who obtained an inequality in the form of the global Lojasiewicz inequality without the denominator [5]. The main approach is to introduce the exponential polynomial function

$$\tau_{-z}(g)(\omega) := g(\omega + z), \tag{53}$$

which satisfies

$$a_j(\tau_{-z}(g)) = a_j(g) e^{\lambda_j z}. \tag{54}$$

If we use $\beta = \{\beta_{j,k}\}$ to represent the set of coefficients in the exponential polynomial (48), the following inequality holds (see [5, pp. 211-214]).

Lemma 5: Let $V = \{z \in \mathbb{C} : g(z) = 0\} \neq \emptyset$ and $\tilde{d}(z, V) = \inf(1, d(z, V))$. Then there exists a positive $\kappa > 0$ independent of β such that $|g(z)| \ge \kappa \|\tau_{-z}(g)\| \tilde{d}(z, V)^{d^o f}$.

We now present the main claim of this subsection, which gives an improved Lojasiewicz inequality to suppress the effect of leading term $|a_j(g)|$.

Proposition 2: Let $V = \{z \in \mathbb{C} : g(z) = 0\} \neq \emptyset$ and $\tilde{d}(z, z_0) = \inf(1, d(z, V))$. Then there exists some $j \in \{1, \ldots, l\}$ and a positive $\kappa > 0$ independent of β such that

$$|g(z)| \ge \kappa \|g\| \frac{\tilde{d}(z, V)^{d^o f}}{(1 + |z|)^{\mu(\mu-1)}} e^{\Re(\lambda_j z)},$$

where $\mu = \max_{1 \le j \le l} m_j - 1$ and κ depends on λ_j and m_j of p_j.

Proof. Observe that from (52), there is a $j \in \{1, \ldots, l\}$ such that

$$\|p_j\| \ge \frac{\|g\|}{l}. \tag{55}$$

Define a positive sequence $\{\gamma_j\}$ of length m_j as follows: Let

$$\eta = (\mu + 1)(1 + |z|)^{2\mu}, \quad \gamma_1 = \gamma = \frac{4\eta \|g\|^2}{l^2 (\mu + 1)(1 + 4\eta)^{\mu+1}}. \tag{56}$$

For $2 \le i \le m_j$, define

$$\gamma_i = 4\eta S_{i-1}, \tag{57}$$

where $S_{i-1}, i \ge 2$ is the partial sum of $\{\gamma_k\}$ $S_{i-1} = \sum_{k=1}^{i-1} \gamma_k$.

It is easy to see that $\{S_i\}$ is a geometric sequence

$$S_i = (1+4\eta)S_{i-1} = \frac{\gamma[(1+4\eta)^i - 1]}{4\eta}. \quad (58)$$

Consequently, there is a $k^* \in \{1, \ldots, m_j\}$ such that

$$|\beta_{j,k^*}|^2 \geq \gamma_{m_j+1-k^*}, \text{ and } |\beta_{j,k}|^2 < \gamma_{m_j+1-k}, k > k^*. \quad (59)$$

Otherwise, by (56) and (58), $\sum_{k=1}^{m_j} |\beta_{j,k}|^2 < S_{m_j} \leq \|f\|^2/(m_j l^2)$, which contradicts to (55) by the fact $\|p_j\| \leq \sqrt{m_j \sum_{k=1}^{m_j} |\beta_{j,k}|^2} < \|g\|/l$.

Let $p_j^* = \sum_{k=1}^{k^*} \beta_{j,k} z^{k-1}$ and $\bar{p}_j^* = p_j - p_j^*$. By (54) and (59), the leading term of $\tau_{-z}(p_j^*)$ satisfies $a(\tau_{-z}(p_j^*)) = \beta_{j,k^*} \neq 0$. So, by Parseval's formula and (59),

$$\frac{1}{2\pi}\int_0^{2\pi} |p_j^*(e^{i\theta}+z)|^2 d\theta \geq |a(\tau_{-z}(p_j^*))|^2 \geq \gamma_{m_j+1-k^*}. \quad (60)$$

Furthermore, from (56) for any $1 \leq k \leq m_j$, $|e^{i\theta} + z|^{2(k-1)} \leq \eta/(\mu+1)$ by (59). One has $|\bar{p}_j^*(e^{i\theta}+z)|^2 \leq m_j \sum_{k=k^*+1}^{m_j} |\beta_{j,k}|^2 |e^{i\theta}+z|^{2(k-1)} \leq \eta S_{m_j-k^*}$. Thus,

$$\frac{1}{2\pi}\int_0^{2\pi} |\bar{p}_j^*(e^{i\theta}+z)|^2 d\theta < \eta S_{m_j-k^*}. \quad (61)$$

Now, $|p_j(e^{i\theta}+z)| \geq |p_j^*(e^{i\theta}+z)| - |\bar{p}_j^*(e^{i\theta}+z)|$, which together with (57), (60) and (61) implies

$$\begin{aligned} &\frac{1}{2\pi}\int_0^{2\pi} |p_j(e^{i\theta}+z)|^2 d\theta \\ &\geq \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{2}|p_j^*(e^{i\theta}+z)|^2 - |\bar{p}_j^*(e^{i\theta}+z)|^2 d\theta \geq \eta S_{m_j-k^*} \end{aligned} \quad (62)$$

where the first inequality follows from $a^2 + c^2 \geq \frac{1}{2}(a+c)^2 \geq \frac{1}{2}b^2$ for nonnegative number a, b, c with $a \geq b - c$.

Note that $\eta > 1$, $(1+4\eta)^i - 1 \geq 4\eta$ for any $i \geq 1$. Then, by (58), (62) and Parseval's formula again, one has $\|\tau_{-z}(p_j)\| \geq \sqrt{\frac{1}{2\pi}\int_0^{2\pi} |p_s(e^{i\theta}+z)|^2 d\theta} \geq \sqrt{\eta\gamma}$, which from (56) immediately yields $\|\tau_{-z}(g)\| \geq \|\tau_{-z}(p_j)\| e^{\Re(\lambda_j z)} \geq \frac{\kappa\|g\|}{(1+|z|)^{\mu(\mu-1)}} e^{\Re(\lambda_j z)}$. Finally, Lemma 5 leads to the claim. □

In this proposition, the leading term $a_j(g)$ is replaced by $\|g\|$. The tradeoff is that κ cannot be explicitly derived, which is a known fact for κ in Grudzinski's inequality [5, pp. 211-214]. Fortunately, although κ cannot be explicitly estimated, from the proof of Proposition 2 and the argument of Grudzinski, κ can be determined by λ_j and m_j, and the vector space of all exponential polynomials $F(z) = \sum_{1\leq j\leq l} p_j(z) e^{\lambda_j z}$ with $\deg p_j \leq m_j$.

B. Numbers of Zeros

The next lemma on the number of zeros of exponential polynomial is also needed in the proof of Theorem 3. It can be obtained from [5, Theorem 3.2.47].

Lemma 6: Let Q be the square $Q := \{z \in \mathbb{C} : a \leq \Re(z) \leq b, c \leq \Im(z) \leq d\}$ and n_Q be the number of zeros of the nontrivial exponential polynomial g defined by (48) in Q. Furthermore, let $\Delta_R := \max_{1\leq i,j\leq l}\{\Im(\lambda_i - \lambda_j)\}$, $\Delta_I := \max_{1\leq i,j\leq l}\{\Re(\lambda_i - \lambda_j)\}$. Then, $n_Q \leq 2d^o g + \frac{1}{2\pi}((b-a)\Delta_R + (c-d)\Delta_I)$.

C. Proof of Theorem 3

The proof is prefaced with some technical lemmas. Let $N > 0$ be a fixed integer and denote $\tau_i = t_i - t_N \in [-t_N, 0]$. The following lemma shows that given $N > 0$, there is a set $V_N \subseteq V$ such that for any $\tau_i \in [-t_N, 0]$, the distance between τ_i and V less than 1 is $|\tau_i - z|$ for some $z \in V_N$, where V is defined in Lemma 5. It also implies that the ratio of cardinality of V_N and t_N is bounded , whose bound is independent of v and N.

Lemma 7: Suppose $V \neq \emptyset$. Let $S_z := \{\tau \in [-t_N, 0] : d(\tau, V) = |\tau - z|\}$, where $z \in V$, and $\Omega_N := \{\tau \in [-t_N, 0] : d(\tau, V) < 1\}$. Then, there is a $V_N = \{z_q, q = 1, \cdots, n_N\} \subset V$ with $n_N = O(t_N + 1)$ such that $\Omega_N \subset \bigcup_{z_q\in V_N} S_{z_q}$, where $V_N = \emptyset$ for $n_N = 0$.

Proof. If $\Omega_N = \emptyset$, take $V_N = \emptyset$ and the lemma is obviously true. Now, suppose $\Omega_N \neq \emptyset$. Let $Q_N = \{z \in \mathbb{C} : -t_N - 1 \leq \Re(z) \leq 1, -1 \leq \Im(z) \leq 1\}$. It is easy to verify that for any $z \notin Q_N$, $d(z, [-t_N, 0]) \geq 1$. So, $Q_N \cap V \neq \emptyset$. Given $v \in \mathbb{C}^n$, let n_N be the number of zeros of f defined by (48) in Q_N. Lemma 6 gives $n_N = O(t_N + 1)$, which is independent of v. As a result, n_N is finite for a given N (noting that n_N may not be uniformly finite for N). Let V_N be the set of all zeros of g in Q_N. Thus, for any $\tau \in \Omega_N$, there is a $z_q \in V_N$ such that $d(\tau, V) = |\tau - z_q|$. This means $\tau \in S_{z_q}$, and hence $\Omega_N \subset \bigcup_{z_q\in V_N} S_{z_q}$. □

Remark 5: The set V_N defined in the proof of Lemma 7 is nondecreasing respect to N. This illustrates that all zeros in V_N are still in V_{N+1}.

Now, let $\widetilde{t}_q = \Re(z_q + t_N)$ for $z_q \in V_N$. Given g, let the number of $\widetilde{t}_q$ in (t_1, t_N) be $L_N - 2$, and hence by Lemma 7, $L_N = O(t_N + 1)$ for all N and v. Without loss of generality, suppose they are $\widetilde{t}_q, q = 2, \cdots, L_N - 1$ arranged in a nondecreasing order. Set $\widetilde{t}_1 = t_1$ and $\widetilde{t}_{L_N} = t_N$ (If $V_N = \emptyset$, there are only two numbers $\widetilde{t}_1$ and $\widetilde{t}_N$). As a consequence, $\widetilde{t}_1 \leq \widetilde{t}_2 \leq \cdots \leq \widetilde{t}_{L_N}$.

Observe that for every $q = 2, \cdots, L_N - 1$, $\widetilde{t}_q \in [\underline{T}, t_N]$ falls into some interval $[t_j, t_{j+1}), 1 \leq j \leq N-1$, let $i_q = j$. Moreover, let $i_1 = 1, i_N = N$. Then, for $2 \leq q \leq L_N$ and $1 \leq i \leq N$, define

$$b_i = \begin{cases} \log i - \log(i_{q-1}+1), & \text{for } \widetilde{t}_{q-1} < t_i \leq \dfrac{\widetilde{t}_{q-1} + \widetilde{t}_q}{2} \\ \log i_q - \log i, & \text{for } \dfrac{\widetilde{t}_{q-1} + \widetilde{t}_q}{2} < t_i \leq \widetilde{t}_q, \end{cases} \quad (63)$$

with $i_1 = 1 \leq i_2 \leq \cdots \leq i_{L_N - 1} \leq N = i_{L_N}$.

Lemma 8: There is a constant $0 < \kappa < 1$ such that the number of b_i that satisfies $b_i \geq \frac{1}{\log N}$ is at least κN for sufficiently large N.

Proof. For any i_q, it is easy to check that if $i \geq e^{\frac{1}{\log N}}(i_{q-1}+1)$ or $i \leq e^{-\frac{1}{\log N}} i_q$, $\log i - \log(i_{q-1}+1) \geq \frac{1}{\log N}$ and $\log i_q - \log i \geq \frac{1}{\log N}$. Hence, for $i_{q-1} < i \leq i_q$, the number of $b_i \geq \dfrac{1}{\log N}$ is larger than $\max\{\lfloor e^{-\frac{1}{\log N}} i_q - e^{\frac{1}{\log N}} i_{q-1} \rfloor - 3, 0\}$ for sufficiently large N such that $e^{\frac{1}{\log N}} < 3$. Consequently, the number of $b_i \geq \dfrac{1}{\log N}$ for $i \in \{1, \ldots, N\}$ satisfies $n_{[1,N]} \geq \sum_{l=2}^{L_N} \max\{\lfloor e^{-\frac{1}{\log N}} i_q - e^{\frac{1}{\log N}} i_{q-1} \rfloor, 0\} - 3(L_N - 1)$. Since $i_1 = 1$ and $e^{\frac{1}{\log N}} i_1 < 3$, one has $\max\left\{\left\lfloor e^{-\frac{1}{\log N}} i_2 - e^{\frac{1}{\log N}} i_1 \right\rfloor, 0\right\} \geq \left\lfloor e^{-\frac{1}{\log N}} i_2 \right\rfloor - 3$, which implies $n_{[1,N]} \geq \sum_{q=3}^{L_N} \max\left\{e^{-\frac{1}{\log N}} i_q - e^{\frac{1}{\log N}} i_{q-1}, 0\right\} + e^{-\frac{1}{\log N}} i_2 - 4L_N$. Note that the right-hand side of the above inequality is minimal when $i_{q-1} = e^{-\frac{2}{\log N}} i_q$, $q = 3, \cdots, L_N$. Therefore, by $L_N = O(\log N)$ and $i_{L_N} = N$, for sufficiently large N, $n_{[1,N]} \geq e^{-\frac{1}{\log N}} i_2 - 4L_N \geq \left(e^{-\frac{2}{\log N}}\right)^{L_N - 1} N - 4L_N \geq \kappa N$, which completes the proof. $\square$

Let $h_i = \min\{1, b_i\}$. The following corollary follows immediately from Lemma 8 for sufficiently large N.

Corollary 3: There is a constant $0 < \kappa < 1$ such that the number of h_i that satisfies $h_i \geq \dfrac{1}{\log N}$ is at least κN.

Lemma 9: Let $V \neq \emptyset$. Then, for any $\tau_i \in [-t_N, 0]$, $\inf\{d(\tau_i, V), 1\} \geq h_i$.

Proof. For any $\tau_i \notin \Omega_N$, $\inf\{d(\tau_i, V), 1\} = 1 \geq h_i$. So,we only need to consider $\tau_i \in \Omega_N$. By Lemma 7, $\Omega_N \subset \bigcup_{z_q \in V_N} S_{z_q}$, that is, for any $\tau_i \in \Omega_N$, there is a $z_q \in V_N$ such that

$$d(\tau_i, V) = |z_q - \tau_i|. \tag{64}$$

Observe that $|z_q - \tau_i| \geq |\Re(z_q - \tau_i)| = |\Re\, z_q - \tau_i| = |\widetilde{t}_q - t_i|$, and $|\widetilde{t}_q - t_i| \geq |\widetilde{t}_1 - t_i|$, for $\widetilde{t}_q \leq t_1$; $|\widetilde{t}_q - t_i| \geq |\widetilde{t}_{L_N} - t_i|$, for $\widetilde{t}_q \geq t_N$. Thus, by (63) and (64), $d(\tau_i, V) \geq b_i$, which is less than the smallest distance between t_i and $\{\widetilde{t}_q, q = 1, \cdots, L_N\}$. Consequently, $\inf\{d(\tau_i, V), 1\} \geq h_i$. $\square$

Proof of Theorem 3: For convenience, we still use the notation Ξ_N in (14) with t_i replaced by $t_i - t_N = \tau_i$. Similar to (12) and (13), $\Phi_N = \Xi_N \Lambda^{-1} W_o$. Since $\Xi_N^* \Xi_N$ is a Hermite matrix, all the eigenvalues are real. By the inequality

$$\sigma_{\min}\left(\Phi_N' \Phi_N\right) \geq \sigma_{\min}\left(W_o' \Lambda^{-*} \Lambda^{-1} W_o\right) \sigma_{\min}\left(\Xi_N^* \Xi_N\right), \tag{65}$$

we only need to estimate $\sigma_{\min}\left(\Xi_N^* \Xi_N\right)$.

By Rayleigh-Ritz theorem $\sigma_{\min}\left(\Xi_N^* \Xi_N\right) = \min_{\|v\|=1, v \in \mathbb{C}^n} v^* \left(\Xi_N^* \Xi_N\right) v$. We now estimate $v^* \Xi_N^* \Xi_N v$ for every $v \in \mathbb{C}^n$ with $\|v\| = 1$. By 14, one has

$$v^* \Xi_N^* \Xi_N v = \sum_{i=1}^{N} |\xi'(\tau_i) v|^2 = \sum_{i=1}^{N} |g(\tau_i)|^2, \tag{66}$$

where g is defined by (48). The following arguments are discussed in two cases.

Case 1: The set $V = \emptyset$ in Lemma 5. Then $g(z) = v e^{\lambda z}$ with $v = \pm 1$. As a result, $|g(\tau_i)| \geq e^{\lambda(\log i - \log N)} = (\dfrac{i}{N})^{\lambda}$, where $\lambda = \max_{1 \leq j \leq l} |\lambda_j|$ is defined by the proposition. Therefore, for sufficiently large N, there is a $\kappa > 0$ such that $\sum_{i=1}^{N} |g(\tau_i)|^2 \geq \dfrac{\sum_{i=1}^{N} i^{2\lambda}}{N^{2\lambda}} \geq \kappa N$, which together with (66), $v^* \Xi_N^* \Xi_N v \geq \kappa N$.

Case 2: $V \neq \emptyset$. By Proposition 2, for some $j \in \{1, \ldots, l\}$, $|g(\tau_i)| \geq \dfrac{\kappa_1 \tilde{d}(\tau_i, z_0)^{\nu} e^{\Re(\lambda_j \tau_i)}}{(1 + |\tau_i|)^{\mu(\mu-1)}}$, where $\nu = d^o f \geq 1$ (If $\nu = 0$, i.e., n=1, f is just an exponential, which belongs to Case 1). By Lemma 9, $\tilde{d}(\tau_i, V) \geq h_i$. Note that $e^{\Re(\lambda_j \tau_i)} \geq e^{-\lambda |\tau_i|}$ for any $j \in \{1, \ldots, l\}$. Consequently, $|g(\tau_i)| \geq \dfrac{\kappa_1 h_i^{\nu} e^{-\lambda |\tau_i|}}{(1 + |\tau_i|)^{\mu(\mu-1)}}$. Hence from Corollary 3 and the fact $\tau_i \geq \log i - \log N$, for sufficiently large N, $\sum_{i=1}^{N} |g(\tau_i)|^2 \geq \dfrac{\kappa_1}{\log^{2[\nu + \mu(\mu-1)]} N} \sum_{i=1}^{\kappa N} (\dfrac{i}{N})^{2\lambda} \geq \dfrac{\kappa_1 N}{\log^{2[\nu + \mu(\mu-1)]} N}$, which together with (65) to (66) completes the proof. $\square$

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